

ZERO KINETIC VISCOSITY- MAGNETIC DIFFUSIVITY LIMIT OF FREE BOUNDARY MAGNETOHYDRODYNAMICS

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ABSTRACT. We consider free boundary magnetohydrodynamics(MHD) in \mathbb{R}^3 . We send ε (kinematic viscosity) and λ (magnetic diffusivity) to zero with same speed. $\varepsilon = \lambda \rightarrow 0$. As like in [1] and [23], we use fixed domain and conormal space (because of boundary layer behavior) to get uniform (in viscosity parameter) regularity and inviscid limit. As like in [1], we encounter bad commutator, so we introduce new variable. Surface tension is not considered and we put zero boundary condition for magnetic field. Finally we get an local unique solution of free boundary MHD without kinetic viscosity and magnetic diffusivity. In the last section we briefly explain how to solve surface tension case. It is very similar to the scheme of [23].

1. INTRODUCTION

In 1981, Beale [5] proved local existence result for free boundary problem of Navier-Stokes equation. Also he proved global regularity for surface tension case in his later work [8]. Similar work was done by many authors. We refer Allain [6] and Tani [7],[16]. In [16], using his local result [7], he showed global existence for both with and without surface tension cases. These works depend on Stokes regularity using diffusive effect of Navier-Stokes. Also look some works by M. Padula and V.A.Solonnikov [17],[18].

Without diffusion, free boundary Euler problem is much harder to solve. For irrotational case, there are more results. Using curl free and divergence free properties of velocity, we can use scalar potential of velocity which solves laplace equation. At result, we can change the problem into a problem on the boundary. We refer some works by S.Wu [9],[10],[11] and recent work by Germain, Masmoudi and Shatah [12] in which they used space-time resonance method. For general case (rotational case), only local results are known. See Lindblad [14]. He used apriori result in [13] and Nash-Moser technique to prove local existence. Also see Coutand and Shkoller [25], and Masmoudi and Rousset [1]. Especially Masmoudi and Rousset [1] solved the problem via inviscid limit.

If we consider conducting fluid (plasma, for instance) we get magnetohydrodynamics. Moving of fluid itself generate electromagnetic force, called Lorentz force. Moreover, for magnetic field, we have Faraday's Law equation, which is a part of Maxwell's equations. Hence we should consider two coupled PDEs with several boundary conditions and divergence free conditions. We refer some results by M. Padula and V.A.Solonnikov [18] with diffusion and apriori estimate result by C.Hao and T.Luo [21] for zero diffusion (for both kinetic and magnetic) When magnetic field has zero boundary condition as like in this paper, local result was gained in [22]. Using this result we will send diffusion parameters to zero in this paper to get inviscid result.

The aim of this paper is to study vanishing 'kinetic viscosity-magnetic diffusivity' limit of free boundary magnetohydrodynamics. Boundary layer behavior make it seems impossible to find a solution in standard sobolev space. Hence we introduce sobolev conormal space which kills normal derivative on the boundary, whereas similar to standard sobolev space away from the boundary. Masmoudi and Rousset used this functional framework to solve vanishing viscosity limit problem of Navier boundary problem [15] and free boundary problem of Navier-Stokes without surface tension [1]. For free boundary with surface tension case, see Tarek and Donghyun [22]. For Free boundary MHD without surface tension, similar bad commutators appear as like in [1] for both two PDEs. We fix this problem using Alinhac's unknown for v and B . If we let $\varepsilon = \lambda$, we can change equations with variables $v + B$ and $v - B$ easily. Also, zero boundary condition of magnetic field helps controlling boundary terms. In [1], (also in this paper) the most challenging part is controlling normal derivative terms. Although we use conormal space, we should control some terms with normal derivatives. We use another variables such as $S_n \doteq \Pi S \mathbf{un}$ or vorticity to control this terms. Nevertheless, by regularity of boundary h , we should give up to get $L^\infty H_{co}^{m-1}$ estimate of $\partial_z v$. Instead, we get $L^4 H_{co}^{m-1}$ regularity of $\partial_z v$ in [1]. Masmoudi and Rousset used Lagrangian coordinate and paradifferential

calculus to get this result. In our problem setting, since magnetic field is zero on the boundary, Lagrangian maps for both $v + B$ and $v - B$ occupy same plasma(fluid) domain. We will use this fact to get similar $L^4 H_{co}^{m-1}$ estimate of $\partial_z v$ and $\partial_z B$. In the last section, we consider surface tension case. We do not provide full detail, but just whole scheme. Since the regularity of h is one more better in surface tension case, whole scheme will be very similar to the work [23].

1.1. Free boundary MHD with Dirichlet magnetic field data. Let ε be kinetic viscosity of Navier-Stokes equation and λ be magnetic diffusivity of Faraday's law. λ is in fact $\lambda = \frac{1}{\mu\sigma}$ where μ is constant vacuum permeability and σ is electric conductivity of material. So, magnetic diffusivity limit means $\sigma \rightarrow \infty$ which implies that plasma becomes perfect conductor.

Let's formulate the problem. Let velocity field $u = (u_1, u_2, u_3)$ and $H = (H_1, H_2, H_3)$. Our domain is where occupied by plasma (or fluid) $\Omega(t)$ and write initial domain as $\Omega(0) \equiv \Omega$. We also write surface as $S_F(t)$, and initial surface as S_F . We consider incompressible viscous-diffusive magnetohydrodynamics (MHD) equations. First equation is Navier-Stokes with Lorentz force and second one is Faraday's law with diffusive effect. Both two vector fields are divergence free, which means that fluid is incompressible and there is no magnetic monopole, which is part of Maxwell's equations. On the boundary, kinematic boundary condition and continuity of stress tensor are considered. We also impose zero magnetic field condition on the boundary.

$$(1.1) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \varepsilon \Delta u + (H \cdot \nabla)H - \frac{1}{2} \nabla |H|^2, & \text{in } \Omega(t) \\ \partial_t H + (u \cdot \nabla)H = \lambda \Delta H + (H \cdot \nabla)u, & \text{in } \Omega(t) \\ \nabla \cdot u = 0, & \text{in } \Omega(t) \\ \nabla \cdot H = 0, & \text{in } \Omega(t) \\ p\mathbf{n} - 2\varepsilon \mathbf{S}(u)\mathbf{n} = gh\mathbf{n} + (H \otimes H - \frac{1}{2}I|H|^2)\mathbf{n}, & \text{on } S_F(t) \\ H = 0, & \text{on } S_F(t) \\ \partial_t h = u^b \cdot \mathbf{N}, & \text{in } S_F(t) \\ u(0) = u_0, \quad H(0) = H_0, \quad \Omega \times \{t = 0\} \end{cases}$$

where $\mathbf{N} = (-\nabla h, 1)$ and $\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$. ε is kinematic viscosity and λ is magnetic diffusivity. Generally $H_{\text{vacuum}} = H_{\text{plasma}}$ is satisfied on the boundary, since nonzero magnetic diffusivity means that plasma is not perfect conductor, which implies zero surface current. Meanwhile, for perfect conductor,

$$H_{\text{vacuum}} \cdot \mathbf{n} = H_{\text{plasma}} \cdot \mathbf{n}$$

by divergence free of H , whereas tangential component can be different, by the existence of surface current. (Maxwell equation) We simply set $H \equiv 0$ condition on the boundary. This is for the case when $|H_{\text{vacuum}}| = 0$. For about pressure, we define total pressure as sum of pressure and magnetic pressure,

$$P = p + \frac{1}{2}|H|^2$$

Without loss of generality, we write P as p . Considering all of these, our system becomes,

$$(1.2) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u - (H \cdot \nabla)H + \nabla p = \varepsilon \Delta u, & \text{in } \Omega(t) \\ \partial_t H + (u \cdot \nabla)H - (H \cdot \nabla)u = \lambda \Delta H, & \text{in } \Omega(t) \\ \nabla \cdot u = 0, & \text{in } \Omega(t) \\ \nabla \cdot H = 0, & \text{in } \Omega(t) \\ p\mathbf{n} - 2\varepsilon \mathbf{S}(u)\mathbf{n} = gh\mathbf{n}, & \text{on } S_F(t) \\ H = 0, & \text{on } S_F(t) \\ \partial_t h = u^b \cdot \mathbf{N}, & \text{in } S_F(t) \\ u(0) = u_0, \quad H(0) = H_0, \quad \Omega \times \{t = 0\} \end{cases}$$

with compatibility conditions

$$(1.3) \quad \begin{cases} \nabla \cdot u_0 = 0 \\ \nabla \cdot H_0 = 0 \\ H_0 = 0, & S_F \\ \Pi \mathbf{S}(u_0)\mathbf{n} = 0 \end{cases}$$

where Π means tangential comonent operator $\Pi = I - \mathbf{n} \otimes \mathbf{n}$, and $\mathbf{S}(u)$ means symmetric part of ∇u ,

$$2\mathbf{S}(u) = (\nabla u) + (\nabla u)^T$$

Note that h is means z component of boundary. This dynamical condition is equivalent to

$$\frac{DF}{Dt} = 0 \quad \text{on} \quad S_F(t)$$

if $S_F(t)$ is represented by the equation $F(x, t) = 0$.

Above system is solved using Sobolev-Slobodetskii space, $W_2^s(\Omega)$ using Lagrangian coordinate transform. (See [22]) In this viscosity limit problem, we use a fixed domain which was used in [1],[22],[3],[4].

1.2. Parametrization to a Fixed domain. We rewrite the problem in the fixed domain (lower half space in \mathbb{R}^3 , $S = \{(x, y, z) | z < 0\}$). This can be done by diffeomorphism $\Phi(t, \cdot)$,

$$\Phi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) \rightarrow \Omega_t$$

$$x = (y, z) \mapsto (y, \varphi(t, y, z)) \in \mathbb{R}^2 \times \mathbb{R}$$

We use function v and q for velocity and pressure on fixed domain S .

$$v(t, x) = u(t, \Phi(t, x)), \quad B(t, x) = H(t, \Phi(t, x)), \quad q(t, x) = p(t, \Phi(t, x))$$

We have to decide $\varphi(t, \cdot)$ so that $\Phi(t, \cdot)$ be a diffeomorphism. (Surely, $\partial_z \varphi \geq 0$, because it is diffeomorphism) There are many ways to take φ . One easy option is to set $\varphi(t, y, z) = z + \eta(t, y)$. But this proper to Euler equations case. Instead of this one, we take a smoothing diffeomorphism as like [1],[3],[4],[22]. This gives $\frac{1}{2}$ better regularity to φ than h .

$$\varphi(t, y, z) = Az + \eta(t, y, z)$$

To ensure that $\Phi(0, \cdot)$ is a diffeomorphism, A should be picked so that

$$\partial_z \varphi(0, y, z) \geq 1, \quad \forall x \in S$$

and η is given by extension of h to domain S , defined by

$$\hat{\eta}(\xi, z) = \chi(z\xi)\hat{h}(\xi)$$

where χ is a smooth, compactly supported function which is 1 on the unit ball $B(0, 1)$. This smoothing diffeomorphism was used in [3],[4],and also in [1]. For this extension, regularity of φ (of course η also.) is $\frac{1}{2}$ better than h . This will be explained in the section2.

We also should define derivative of v, B in S , so that measure $\partial_i u, \partial_i B$ in Φ_t . Then we rewrite our system (1.2) and (1.3) in a fixed domain S . Using change of variable, we get

$$(\partial_i u)(t, y, \phi) = (\partial_i v - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z v)(t, y, z) \quad i = t, 1, 2$$

$$(\partial_i H)(t, y, \phi) = (\partial_i B - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z B)(t, y, z) \quad i = t, 1, 2$$

$$(\partial_3 u)(t, y, \phi) = (\frac{1}{\partial_z \varphi} \partial_z v)(t, y, z)$$

$$(\partial_3 H)(t, y, \phi) = (\frac{1}{\partial_z \varphi} \partial_z B)(t, y, z)$$

So we define the following operator as like in [1].

$$\partial_i^\varphi = \partial_i - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z \quad i = t, 1, 2, \quad \partial_3^\varphi = \frac{1}{\partial_z \varphi} \partial_z$$

This definition implies that $\partial_i u \circ \Phi = \partial_i^\varphi v$, $i = t, 1, 2, 3$, and similarly to magnetic field. Hence our system in S becomes,

$$(1.4) \quad \begin{cases} \partial_t^\varphi v + v \cdot \nabla^\varphi v - B \cdot \nabla^\varphi B + \nabla^\varphi q = 2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi v), & \text{in } S \\ \partial_t^\varphi B + v \cdot \nabla^\varphi B - B \cdot \nabla^\varphi v = 2\lambda \nabla^\varphi \cdot (\mathbf{S}^\varphi B), & \text{in } S \\ \nabla^\varphi \cdot v = 0, & \text{in } S \\ \nabla^\varphi \cdot B = 0, & \text{in } S \\ q^b \mathbf{n} - 2\varepsilon (\mathbf{S}^\varphi v) \mathbf{n} = gh \mathbf{n}, & \text{on } \partial S \\ \partial_t h = v \cdot \mathbf{N}, & \text{on } \partial S \\ B = 0, & \text{on } \partial S \\ v(0) = v_0, \quad B(0) = B_0, & \text{in } S \end{cases}$$

with compatibility condition

$$(1.5) \quad \begin{cases} \nabla^\varphi \cdot v_0 = 0, & \text{in } S \\ \nabla^\varphi \cdot B_0 = 0, & \text{in } S \\ B_0 = 0, & \text{on } \partial S \\ \Pi \mathbf{S}^\varphi(v_0) \mathbf{n} = 0, & \text{on } \partial S \end{cases}$$

1.3. Functional Framework and Notations. We briefly introduce conormal space and some function space that is proper to our problem. See [1] for more explanation. First we define sobolev conormal derivatives.

Definition 1.1. We define the followings.

$$Z_1 = \partial_x, \quad Z_2 = \partial_y, \quad Z_3 = \frac{z}{1-z} \partial_z, \quad Z^\alpha = Z^{(\alpha_1, \alpha_2, \alpha_3)}$$

$$|f|_{H_{co}^s(S)}^2 \doteq \sum_{|\alpha| \leq s} |Z^\alpha f|_{L^2(S)}^2, \quad |f|_{W_{co}^{s,\infty}(S)} \doteq \sum_{|\alpha| \leq s} |Z^\alpha f|_{L^\infty(S)}$$

In this paper we abbreviate the notation as $|\cdot|_s = |\cdot|_{H^s}$, $\|\cdot\|_s = \|\cdot\|_{H_{co}^s}$ and $\|\cdot\| = \|\cdot\|_{L^2}$. And similarly $|\cdot|_{s,\infty} = |\cdot|_{W^{s,\infty}}$ and $\|\cdot\|_{s,\infty} = \|\cdot\|_{W_{co}^{s,\infty}}$. And for horizontal component, we use $v_y \equiv (v_1, v_2)$ and $\nabla_y \equiv (\partial_1, \partial_2)$. Sometimes we may use notation Z^m . This means some Z^α where $|\alpha| = m$. We will add for all such α , so don't need to specify α .

Definition 1.2. For $m \geq 1$,

$$E^m = \{f \in H_{co}^m, \partial_z f \in H_{co}^{m-1}\}, \quad E^{m,\infty} = \{f \in W_{co}^{m,\infty}, \partial_z f \in W_{co}^{m-1,\infty}\}$$

with norms,

$$\|f\|_{E^m}^2 = \|f\|_m^2 + \|\partial_z f\|_{m-1}^2, \quad \|f\|_{E^{m,\infty}} = \|f\|_{m,\infty} + \|\partial_z f\|_{m-1,\infty}$$

We also define tangential spaces.

Definition 1.3.

$$H_{tan}^s(S) = \{f \in L^2(S), \Lambda^s f \in L^2(S)\}$$

where Λ^s is tangential Fourier multiplier by $(1 + |\xi|^2)^{s/2}$, with norm

$$\|f\|_{H_{tan}^s} = \|\Lambda^s f\|_{L^2}$$

Notation In this paper, Λ_0 means $\Lambda(\frac{1}{c_0})$ where $\Lambda(\cdot)$ is a monotone increasing function and $\Lambda(\cdot, \cdot)$ is a monotone increasing function with all its variables. They may vary line to line.

1.4. Main results. We state two main results for this paper. First theorem is about local existence in uniform time in Sobolev conormal space.

Theorem 1.4. *For fixed sufficiently large $m, (m \geq 6)$, let initial data be given so that*

$$\sup_{\varepsilon \in (0,1]} (|h_0^\varepsilon|_m + \sqrt{\varepsilon}|h_0^\varepsilon|_{m+\frac{1}{2}} + \|v_0^\varepsilon\|_m + \|B_0^\varepsilon\|_m + \|\partial_z v_0^\varepsilon\|_{m-1} + \|\partial_z B_0^\varepsilon\|_{m-1})$$

$$(1.6) \quad + \|\partial_z v_0^\varepsilon\|_{1,\infty} + \|\partial_z B_0^\varepsilon\|_{1,\infty} + \sqrt{\varepsilon}\|\partial_{zz} v_0^\varepsilon\|_{L^\infty} + \sqrt{\varepsilon}\|\partial_{zz} B_0^\varepsilon\|_{L^\infty}) \leq R$$

and satisfy compatibility conditions

$$(1.7) \quad \nabla^\varphi \cdot v_0^\varepsilon = 0, \quad \nabla^\varphi \cdot B_0^\varepsilon = 0, \quad \text{in } S$$

$$\Pi(S^\varphi v_0^\varepsilon) \hat{n} = 0, \quad B_0^\varepsilon = 0 \quad \Pi \doteq I - \hat{n} \otimes \hat{n}, \quad \text{on } \partial S$$

Then for $\forall \varepsilon \in (0, 1]$, there exist $T > 0$ (independent to ε), and some $C > 0$, such that there exist a unique solution $(v^\varepsilon, B^\varepsilon, h^\varepsilon)$ for (1.4) with $\varepsilon = \lambda$ on $[0, T]$, and the following energy estimates hold. Non dissipative energy,

$$(1.8) \quad \sup_{t \in [0, T]} (|h^\varepsilon|_m^2 + \|v^\varepsilon\|_m^2 + \|B^\varepsilon\|_m^2 + \|\partial_z v^\varepsilon\|_m^2 + \|\partial_z B^\varepsilon\|_m^2 + \|\partial_z v^\varepsilon\|_{1,\infty}^2 + \|\partial_z B^\varepsilon\|_{1,\infty}^2)$$

$$+ \|\partial_z v^\varepsilon\|_{L^4([0, T], H_{co}^{m-1})}^2 + \|\partial_z B^\varepsilon\|_{L^4([0, T], H_{co}^{m-1})}^2 \leq C$$

Dissipative terms,

$$(1.9) \quad \sup_{t \in [0, T]} \left(\varepsilon |h^\varepsilon|_{m+\frac{1}{2}}^2 + \varepsilon \|\partial_{zz} v^\varepsilon\|_{L^\infty}^2 + \varepsilon \|\partial_{zz} B^\varepsilon\|_{L^\infty}^2 \right)$$

$$+ \varepsilon \int_0^T (\|\nabla v^\varepsilon\|_m^2 + \|\nabla B^\varepsilon\|_m^2 + \|\nabla \partial_z v^\varepsilon\|_{m-2}^2 + \|\nabla \partial_z B^\varepsilon\|_{m-2}^2) \leq C$$

We get zero kinetic viscosity-magnetic diffusivity limit using the result of theorem 1.4

Theorem 1.5. *Let's assume that we*

$$(1.10) \quad \lim_{\varepsilon \rightarrow 0} \left(\|v_0^\varepsilon - v_0\|_{L^2(S)} + \|B_0^\varepsilon - B_0\|_{L^2(S)} + \|h_0^\varepsilon - h_0\|_{L^2(\partial S)} \right) = 0$$

where $(v_0^\varepsilon, B_0^\varepsilon, h_0^\varepsilon)$ and (v_0, B_0, h_0) satisfy assumption of Theorem 1.4. Then there exist (v, B, h) such that those are in

$$(1.11) \quad v, B \in L^\infty([0, T], H_{co}^m(S)), \quad \partial_z v, \partial_z B \in L^\infty([0, T], H_{co}^{m-1}(S)), \quad h \in L^\infty([0, T], H_{co}^m(\mathbb{R}^2))$$

and

$$(1.12) \quad \lim_{\varepsilon \rightarrow 0} \sup_{[0, T]} (\|v^\varepsilon - v\|_{L^2(S)} + \|v^\varepsilon - v\|_{L^\infty(S)})$$

$$+ \|B^\varepsilon - B\|_{L^2(S)} + \|B^\varepsilon - B\|_{L^\infty(S)} + \|h^\varepsilon - h\|_{L^2(\partial S)} + \|h^\varepsilon - h\|_{W^{1,\infty}(\partial S)} = 0$$

This limit solves free boundary inviscid magnetohydrodynamics equation.

$$(1.13) \quad \begin{cases} \partial_t^\varphi v + v \cdot \nabla^\varphi v + \nabla^\varphi q = B \cdot \nabla^\varphi B, & \text{in } S \\ \partial_t^\varphi B + v \cdot \nabla^\varphi B = B \cdot \nabla^\varphi v, & \text{in } S \\ \nabla^\varphi \cdot v = 0, & \text{in } S \\ \nabla^\varphi \cdot B = 0, & \text{in } S \\ q^b \mathbf{n} = gh \mathbf{n}, & \text{on } \partial S \\ \partial_t h = v \cdot \mathbf{N}, & \text{on } \partial S \\ B = 0, & \text{on } \partial S \\ v(0) = v_0, \quad B(0) = B_0, & \text{in } S \end{cases}$$

2. FORMAL DIFFERENTIATION AND ALINHAC'S UNKNOWN

Suppose that we have a smooth solution (v, B, h) . Standard energy estimate gives (see energy estimates in section 6,7) that regularities of (v, B) and h are same order. Then commutators generate some terms that are not controlled by (v, B, h) . For example,

$$\partial_t^\varphi + v \cdot \nabla^\varphi = \partial_t + v_y \nabla_y + \frac{1}{\partial_z \varphi} (v \cdot \mathbf{N} - \partial_t \eta) \partial_z, \quad \mathbf{N} = (-\nabla_y \varphi, 1)$$

If we apply conormal derivative Z^α , ($|\alpha| = m$) then we encounter some commutators look like $\|Z^\alpha \mathbf{N}\| \sim |\nabla \varphi|_m \sim |\varphi|_{m+1} \sim |h|_{m+\frac{1}{2}}$ (since φ is $\frac{1}{2}$ smoother than h by chosen diffeomorphism. See [1],[22], or section 4 for this estimate.) Hence, as introduced in [1], we rewrite the system in terms of 'Alinhac's unknown', because this new unknown kills all bad commutators. Note that when surface tension exists, h is smoother than v, B by one, so $|h|_{m+\frac{1}{2}}$ is not bad commutator anymore. Nevertheless, surface tension terms generate other bad commutators like $|h|_{m+\frac{3}{2}}$, which means $\frac{1}{2}$ lack of regularity. See [22] for more detail. Anyway, for both cases, standard energy estimates lack of $\frac{1}{2}$ order for the regularity of h . When surface tension does not exist, we can use Alinhac's unknown whereas we use time derivative for the surface tension case. In this paper we make two Alinhac's unknown for both v and B . In this section, we systemically construct Alinhac's unknown. First, let's define the following

$$(2.1) \quad \begin{cases} \mathcal{N}(v, B, q, \varphi) = \partial_t^\varphi v + (v \cdot \nabla^\varphi) v + \nabla^\varphi q - 2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi v) - (B \cdot \nabla^\varphi) B \\ \mathcal{F}(v, B, \varphi) = \partial_t^\varphi B + (v \cdot \nabla^\varphi) B - (B \cdot \nabla^\varphi) v - 2\lambda \nabla^\varphi \cdot (\mathbf{S}^\varphi B) \\ d_v(v, \varphi) = \nabla^\varphi \cdot v \\ d_B(B, \varphi) = \nabla^\varphi \cdot B \\ \mathcal{B}(v, B, q, \varphi) = (q - gh)\mathbf{N} - 2\varepsilon(\mathbf{S}^\varphi v)\mathbf{N} \end{cases}$$

In the following, \sim means first order expansion for \dot{f}, \dot{g} .

Proposition 2.1. *We have the following first order expansion.*

$$(f + \dot{f}) \cdot \nabla^{\varphi+\dot{\varphi}}(g + \dot{g}) \sim (f \cdot \nabla^\varphi)g + (f \cdot \nabla^\varphi)\dot{g} + (\dot{f} \cdot \nabla^\varphi)g - \partial_z^\varphi g(f \cdot (\nabla^\varphi \dot{\varphi}))$$

Proof. Index i means summation over $i = 1, 2$.

$$\begin{aligned} (f + \dot{f}) \cdot \nabla^{\varphi+\dot{\varphi}}(g + \dot{g}) &= (f_i + \dot{f}_i) \left(\partial_i g + \partial_i \dot{g} - \frac{\partial_i \varphi + \partial_i \dot{\varphi}}{\partial_z \varphi + \partial_z \dot{\varphi}} (\partial_z g + \partial_z \dot{g}) \right) + (f_3 + \dot{f}_3) \frac{\partial_z g + \partial_z \dot{g}}{\partial_z \varphi + \partial_z \dot{\varphi}} \\ &\sim f_i \left(\partial_i g + \partial_i \dot{g} - \frac{(\partial_i \varphi + \partial_i \dot{\varphi})(\partial_z g + \partial_z \dot{g})}{\partial_z \varphi} \left(1 - \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \right) \right) \\ &\quad + \dot{f}_i \left(\partial_i g + \partial_i \dot{g} - \frac{(\partial_i \varphi + \partial_i \dot{\varphi})(\partial_z g + \partial_z \dot{g})}{\partial_z \varphi} \right) + (f_3 + \dot{f}_3) \frac{\partial_z g + \partial_z \dot{g}}{\partial_z \varphi + \partial_z \dot{\varphi}} \\ &\sim f_i \partial_i g + f_i \partial_i \dot{g} - f_i \frac{(\partial_i \varphi \partial_z g + \partial_i \varphi \partial_z \dot{g} + \partial_i \dot{\varphi} \partial_z g)}{\partial_z \varphi} \left(1 - \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \right) \\ &\quad + \dot{f}_i \partial_i g - \frac{\dot{f}_i}{\partial_z \varphi} \partial_i \varphi \partial_z g + \frac{g_3}{\partial_z \varphi} \left(\partial_z g + \partial_z \dot{g} - \partial_z g \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \right) + \frac{\dot{f}_3}{\partial_z \varphi} \partial_z g \\ &\sim f_i \partial_i^\varphi g + f_i \partial_i^\varphi \dot{g} + \dot{f}_i \partial_i^\varphi g - g_i \partial_i \dot{\varphi} + \dot{f}_i \frac{\partial_i \varphi \partial_z g}{\partial_z \varphi} \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \\ &\quad + f_3 \partial_3^\varphi g + f_3 \partial_3^\varphi \dot{g} + \dot{f}_3 \partial_3^\varphi g - f_3 \frac{\partial_z g \partial_z \dot{\varphi}}{\partial_z \varphi \partial_z \varphi} \\ &\sim f \cdot \nabla^\varphi g + f \cdot \nabla^\varphi \dot{g} + \dot{f} \cdot \nabla^\varphi g - \frac{\partial_z g}{\partial_z \varphi} (f_i \partial_i^\varphi \dot{\varphi} + f_3 \partial_3^\varphi \dot{\varphi}) \\ &= (f \cdot \nabla^\varphi)g + (f \cdot \nabla^\varphi)\dot{g} + (\dot{f} \cdot \nabla^\varphi)g - \partial_z^\varphi g(f \cdot (\nabla^\varphi \dot{\varphi})) \end{aligned}$$

□

Proposition 2.2. *We have the following first order expansion.*

$$\partial_i^{\varphi+\dot{\varphi}} |f + \dot{f}|^2 \sim \partial_i^\varphi |f|^2 + 2\dot{f} \cdot (\partial_i^\varphi f) + 2f \cdot (\partial_i^\varphi \dot{f}) - 2\partial_i^\varphi \dot{\varphi} (f \cdot \partial_z^\varphi f)$$

Proof. Index j means summation over $j = 1, 2, 3$. For $i = 1, 2$,

$$\begin{aligned}
\partial_i^{\varphi+\dot{\varphi}}|f + \dot{f}|^2 &\sim \left(\partial_i - \frac{\partial_i \varphi + \partial_i \dot{\varphi}}{\partial_z \varphi + \partial_z \dot{\varphi}} \partial_z \right) (f_j^2 + 2f_j \dot{f}_j) \\
&\sim 2f_j \partial_i f_j + 2\partial_i f_j \dot{f}_j + 2f_j \partial_i \dot{f}_j - 2 \frac{(\partial_i \varphi + \partial_i \dot{\varphi})(f_j \partial_z f_j + \partial_z f_j \dot{f}_j + f_j \partial_z \dot{f}_j)}{\partial_z \varphi} \left(1 - \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \right) \\
&\sim \partial_i |f_j|^2 + 2\dot{f}_j \partial_i f_j + 2f_j \partial_i \dot{f}_j - \frac{2}{\partial_z \varphi} (\partial_i \varphi f_j \partial_z f_j + \partial_i \varphi \partial_z f_j \dot{f}_j + \partial_i \varphi f_j \partial_z \dot{f}_j + \partial_i \dot{\varphi} f_j \partial_z f_j) \\
&\quad + \frac{2}{\partial_z \varphi} \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \partial_i \varphi f_j \partial_z f_j \\
&\sim \left(\partial_i |f_j|^2 - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z |f_j|^2 \right) + 2\dot{f}_j \left(\partial_i f_j - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z f_j \right) + 2f_j \left(\partial_i \dot{f}_j - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z \dot{f}_j \right) \\
&\quad - 2f_j \frac{\partial_z f_j}{\partial_z \varphi} \left(\partial_i \dot{\varphi} - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z \dot{\varphi} \right) \\
&= \partial_i^{\varphi} |f|^2 + 2\dot{f} \cdot (\partial_i^{\varphi} f) + 2f \cdot (\partial_i^{\varphi} \dot{f}) - 2\partial_i^{\varphi} \dot{\varphi} (f \cdot \partial_z^{\varphi} f)
\end{aligned}$$

For $i = 3$,

$$\begin{aligned}
\partial_3^{\varphi+\dot{\varphi}}|f + \dot{f}|^2 &\sim \frac{1}{\partial_z \varphi} \left(1 - \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \right) \partial_z (f_j^2 + 2f_j \dot{f}_j) \\
&\sim \frac{2}{\partial_z \varphi} (f_j \partial_z f_j + \partial_z f_j \dot{f}_j + f_j \partial_z \dot{f}_j) \left(1 - \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} \right) \\
&\sim \frac{2}{\partial_z \varphi} \left(f_j \partial_z f_j + \partial_z f_j \dot{f}_j + f_j \partial_z \dot{f}_j - \frac{\partial_z \dot{\varphi}}{\partial_z \varphi} f_j \partial_z f_j \right) \\
&= \partial_3^{\varphi} |f|^2 + 2\dot{f} \cdot (\partial_3^{\varphi} f) + 2f \cdot (\partial_3^{\varphi} \dot{f}) - 2\partial_3^{\varphi} \dot{\varphi} (f \cdot \partial_z^{\varphi} f)
\end{aligned}$$

□

Using proposition 2.1 and 2.2, we get linearization for (2.1),

$$\begin{aligned}
DN(v, B, q, \varphi) \cdot (\dot{v}, \dot{B}, \dot{q}, \dot{\varphi}) &= \partial_t^{\varphi} \dot{v} + (v \cdot \nabla^{\varphi}) \dot{v} + \nabla^{\varphi} \dot{q} - 2\varepsilon \nabla^{\varphi} \cdot (\mathbf{S}^{\varphi} \dot{v}) - (B \cdot \nabla^{\varphi}) \dot{B} \\
&\quad + (\dot{v} \cdot \nabla^{\varphi}) v - \partial_z^{\varphi} v (\partial_t^{\varphi} \dot{\varphi} + v \cdot \nabla^{\varphi} \dot{\varphi}) - \partial_z^{\varphi} q \nabla^{\varphi} \dot{\varphi} \\
&\quad + 2\varepsilon \nabla^{\varphi} (\partial_z^{\varphi} v \otimes \nabla^{\varphi} \dot{\varphi} + \nabla^{\varphi} \dot{\varphi} \otimes \partial_z^{\varphi} v) + 2\varepsilon \partial_z^{\varphi} (\mathbf{S}^{\varphi} v) \nabla^{\varphi} \dot{\varphi} \\
&\quad - \left(\dot{B} \cdot \nabla^{\varphi} B - \partial_z^{\varphi} B (B \cdot \nabla^{\varphi} \dot{\varphi}) \right) \\
Dd_v(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) &= \nabla^{\varphi} \cdot \dot{v} - \nabla^{\varphi} \dot{\varphi} \cdot \partial_z^{\varphi} v \\
Dd_B(B, \varphi) \cdot (\dot{B}, \dot{\varphi}) &= \nabla^{\varphi} \cdot \dot{B} - \nabla^{\varphi} \dot{\varphi} \cdot \partial_z^{\varphi} B \\
DF(v, B, \varphi) \cdot (\dot{v}, \dot{B}, \dot{\varphi}) &= \partial_t^{\varphi} \dot{B} + (v \cdot \nabla^{\varphi}) \dot{B} - (B \cdot \nabla^{\varphi}) \dot{v} - 2\lambda \nabla^{\varphi} \cdot (\mathbf{S}^{\varphi} \dot{B}) \\
&\quad + (\dot{v} \cdot \nabla^{\varphi}) B - (\dot{B} \cdot \nabla^{\varphi}) v - \partial_z^{\varphi} B (\partial_t^{\varphi} \dot{\varphi} + v \cdot \nabla^{\varphi} \dot{\varphi}) + \partial_z^{\varphi} v (B \cdot \nabla^{\varphi} \dot{\varphi}) \\
&\quad + 2\lambda \nabla^{\varphi} (\partial_z^{\varphi} B \otimes \nabla^{\varphi} \dot{\varphi} + \nabla^{\varphi} \dot{\varphi} \otimes \partial_z^{\varphi} B) + 2\lambda \partial_z^{\varphi} (\mathbf{S}^{\varphi} B) \nabla^{\varphi} \dot{\varphi} \\
DB(v, B, q, \varphi) \cdot (\dot{v}, \dot{B}, \dot{q}, \dot{\varphi}) &= 2\varepsilon \mathbf{S}^{\varphi} \dot{v} \mathbf{N} - \partial_z^{\varphi} v \otimes \nabla^{\varphi} \dot{\varphi} \mathbf{N} - \nabla^{\varphi} \dot{\varphi} \otimes \partial_z^{\varphi} v \mathbf{N} - (\dot{q} - g\dot{h}) \mathbf{N} \\
&\quad + (2\varepsilon \mathbf{S}^{\varphi} v - (q - gh)) \dot{\mathbf{N}}
\end{aligned}$$

On the right hand side of above linearizations, we see that $\nabla^{\varphi} \varphi$, which look like $|\nabla \varphi|_m \sim |h|_{m+\frac{1}{2}}$ for high order estimate. Now, we define Alinhac's new unknowns to remove $\nabla^{\varphi} \varphi$'s on the right hand side. For example, on the right hand side of \mathcal{N} , we see that $-(\partial_z^{\varphi} v) v \cdot \nabla^{\varphi} \dot{\varphi}$ is one of bad terms. But this term has $v \cdot \nabla^{\varphi}$ so this can be combined nonlinear with term $(v \cdot \nabla^{\varphi}) \dot{v}$. *i.e.*, $-(\partial_z^{\varphi} v) v \cdot \nabla^{\varphi} \dot{\varphi}$ gives 1 derivative ∇^{φ} to nonlinear structure $(v \cdot \nabla^{\varphi})$ and remained $-(\partial_z^{\varphi} v) \dot{\varphi}$ is combined with \dot{v} to generate a new variable.

$$(2.2) \quad \mathcal{V} \doteq \dot{v} - \partial_z^{\varphi} v \dot{\varphi}, \quad \mathcal{Q} \doteq \dot{q} - \partial_z^{\varphi} q \dot{\varphi}, \quad \mathcal{B} \doteq \dot{B} - \partial_z^{\varphi} B \dot{\varphi}$$

Lemma 2.3. *Let's define*

$$\mathcal{A}_i(v, \varphi) = \partial_i^\varphi v, \quad \mathcal{F}_{ij}(v, \varphi) = \partial_i^\varphi \partial_j^\varphi v$$

then linearization of \mathcal{A}, \mathcal{F} can be expressed by

$$D\mathcal{A}_i(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \partial_i^\varphi (\dot{v} - \partial_z^\varphi v \dot{\varphi}) + \dot{\varphi} \partial_z^\varphi (\mathcal{A}_i(v, \varphi))$$

$$D\mathcal{F}_{ij}(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) = \partial_{ij}^\varphi (\dot{v} - \partial_z^\varphi v \dot{\varphi}) + \dot{\varphi} \partial_z^\varphi (\mathcal{F}_{ij}(v, \varphi))$$

Proof. This is simple calculations which using commutative property of ∂_i^φ . See [1] proof of Lemma 2.7 for detail. \square

Using Lemma 2.3, we have the following proposition. Note that on the right hand side, all bad terms (which looks like $|h|_{m+\frac{1}{2}}$) are removed.

Proposition 2.4. *Linearization of (2.1) can be expressed as the following, using new unknowns $\mathcal{V}, \mathcal{Q}, \mathcal{B}$.*

$$\begin{aligned} DN(v, B, q, \varphi) \cdot (\dot{v}, \dot{B}, \dot{q}, \dot{\varphi}) &= (\partial_t^\varphi + (v \cdot \nabla^\varphi) - 2\varepsilon \nabla \cdot (\mathbf{S}^\varphi)) \mathcal{V} + \nabla^\varphi \mathcal{Q} - (B \cdot \nabla^\varphi) \mathcal{B} \\ &\quad + (\dot{v} \cdot \nabla^\varphi) v - (\dot{B} \cdot \nabla^\varphi) B + \dot{\varphi} \{ \partial_z^\varphi \mathcal{N}(v, B, P, \varphi) - (\partial_z^\varphi v \cdot \nabla^\varphi) v + (\partial_z^\varphi B \cdot \nabla^\varphi) B \} \\ Dd_v(v, \varphi) \cdot (\dot{v}, \dot{\varphi}) &= \nabla^\varphi \cdot \mathcal{V} - \dot{\varphi} \partial_z^\varphi (d_v(v, \varphi)) \\ Dd_B(B, \varphi) \cdot (\dot{B}, \dot{\varphi}) &= \nabla^\varphi \cdot \mathcal{B} - \dot{\varphi} \partial_z^\varphi (d_B(B, \varphi)) \\ D\mathcal{F}(v, B, \varphi) \cdot (\dot{v}, \dot{B}, \dot{\varphi}) &= (\partial_t^\varphi + (v \cdot \nabla^\varphi) - 2\lambda \nabla \cdot (\mathbf{S}^\varphi)) \mathcal{B} - (B \cdot \nabla^\varphi) \mathcal{V} \\ &\quad + (\dot{v} \cdot \nabla^\varphi) B - (\dot{B} \cdot \nabla^\varphi) v + \dot{\varphi} \{ \partial_z^\varphi (\mathcal{F}(v, B, \varphi)) - (\partial_z^\varphi v \cdot \nabla^\varphi) B + (\partial_z^\varphi B \cdot \nabla^\varphi) v \} \\ D\mathcal{B}(v, B, q, \varphi) \cdot (\dot{v}, \dot{B}, \dot{q}, \dot{\varphi}) &= 2\varepsilon \mathbf{S}^\varphi \mathcal{V} \mathbf{N} + 2\varepsilon \dot{\varphi} \partial_z (\mathbf{S}^\varphi v) \mathbf{N} - (\dot{q} - g\dot{h}) \mathbf{N} + (2\varepsilon \mathbf{S}^\varphi - (q - gh)) \dot{\mathbf{N}} \end{aligned}$$

Proof. Use linearization of (2.1) and Lemma 2.3. \square

3. PRELIMINARIES ESTIMATES

In this section we just collect some necessary propositions and preliminary estimates from [1].

Proposition 3.1. *We have the following products, and commutator estimates.*

- For $u, v \in L^\infty \cap H_{co}^k$, $k \geq 0$,

$$(3.1) \quad \|Z^{\alpha_1} u Z^{\alpha_2} v\| \lesssim \|u\|_{L^\infty} \|v\|_k + \|v\|_{L^\infty} \|u\|_k, \quad |\alpha_1| + |\alpha_2| = k$$

- For $1 \leq |\alpha| \leq k$, $g \in H_{co}^{k-1} \cap L^\infty$, $f \in H_{co}^k$ such that $Zf \in L^\infty$, we have

$$(3.2) \quad \|[Z^\alpha, f]g\| \lesssim \|Zf\|_{k-1} \|g\|_{L^\infty} + \|Zf\|_{L^\infty} \|g\|_{k-1}$$

- For $|\alpha| = k \geq 2$, we define the symmetric commutator $[Z^\alpha, f, g] = Z^\alpha(fg) - (Z^\alpha f)g - fZ^\alpha g$. Then we have the estimate

$$(3.3) \quad \|[Z^\alpha, f, g]\| \lesssim \|Zf\|_{L^\infty} \|Zg\|_{k-2} + \|Zg\|_{L^\infty} \|Zf\|_{k-2}$$

The followings are embedding and trace estimate.

Proposition 3.2. • For $s_1 \geq 0, s_2 \geq 0$ such that $s_1 + s_2 > 2$ and f such that $f \in H_{tan}^{s_1}$, $\partial_z f \in H_{tan}^{s_2}$, we have the anisotropic sobolev embedding.

$$(3.4) \quad \|f\|_{L^\infty}^2 \lesssim \|\partial_z f\|_{H_{tan}^{s_2}} \|f\|_{H_{tan}^{s_1}}$$

- For $f \in H^1(S)$, we have the trace estimates,

$$(3.5) \quad |f(\cdot, 0)|_{H^s(\mathbb{R}^2)} \leq C \|\partial_z f\|_{H_{tan}^{s_2}}^{1/2} \|f\|_{H_{tan}^{s_1}}^{1/2}$$

with $s_1 + s_2 = 2s \geq 0$.

Similar estimates on \mathbb{R}^2 .

Proposition 3.3.

$$(3.6) \quad |\Lambda^s(fg)|_{L^2(\mathbb{R}^2)} \leq C_s (|f|_{L^\infty(\mathbb{R}^2)}|g|_{H^s(\mathbb{R}^2)} + |g|_{L^\infty(\mathbb{R}^2)}|f|_{H^s(\mathbb{R}^2)})$$

$$(3.7) \quad \|[\Lambda^s, f]\nabla g\|_{L^2(\mathbb{R}^2)} \leq C_s (|\nabla f|_{L^\infty(\mathbb{R}^2)}|g|_{H^s(\mathbb{R}^2)} + |\nabla g|_{L^\infty(\mathbb{R}^2)}|f|_{H^s(\mathbb{R}^2)})$$

$$(3.8) \quad |uv|_{\frac{1}{2}} \lesssim |u|_{1,\infty}|v|_{\frac{1}{2}}$$

Jacobian of change of variable Φ is $\partial_z \varphi$. Let

$$\begin{aligned} F(t, \Phi(t, y, z)) &= f(t, y, z) \\ \int_{\Omega(t)} F dy dz &= \int_S f(\partial_z \varphi) dy dz \doteq \int_S f dV_t \end{aligned}$$

Now we state integration by part for $\int_S \partial_i^\varphi f g dV_t$.

Proposition 3.4.

$$(3.9) \quad \int_S \partial_i^\varphi f g dV_t = - \int_S f \partial_i^\varphi g dV_t + \int_{\partial S} f g \mathbf{N}_i dy, \quad i = 1, 2, 3$$

$$(3.10) \quad \int_S \partial_t^\varphi f g dV_t = \partial_t \int_S f g dV_t - \int_S f \partial_t^\varphi g dV_t - \int_{\partial S} f g \partial_t h$$

Corollary 3.5. *Let v be a vector field such that $\nabla^\varphi \cdot v = 0$. For every smooth function f, g and smooth vector field u, w , we have*

$$(3.11) \quad \int_S (\partial_t f + v \cdot \nabla^\varphi f) f dV_t = \frac{1}{2} \partial_t \int_S |f|^2 dV_t - \frac{1}{2} \int_{\partial S} |f|^2 (\partial_t h - v \cdot \mathbf{N}) dy$$

$$(3.12) \quad \int_S \Delta^\varphi f g dV_t = - \int_S \nabla^\varphi f \cdot \nabla^\varphi g dV_t + \int_{\partial S} \nabla^\varphi f \cdot \mathbf{N} g dy$$

$$(3.13) \quad \int_S \nabla^\varphi \cdot (\mathbf{S}^\varphi u) \cdot w dV_t = - \int_S \mathbf{S}^\varphi u \cdot \mathbf{S}^\varphi w dV_t + \int_{\partial S} (\mathbf{S}^\varphi u \mathbf{N}) \cdot w dy$$

Adapted Korn's inequality in S .

Proposition 3.6. *Let $\partial_z \varphi \geq c_0$, $\|\nabla \varphi\|_{L^\infty} + \|\nabla^2 \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then there exists $\Lambda_0 = \Lambda(1/c_0) > 0$, such that for every $v \in H^1(S)$, we have*

$$(3.14) \quad \|\nabla v\|_{L^2(S)}^2 \leq \Lambda_0 \left(\int_S |\mathbf{S}^\varphi v|^2 dV_t + \|v\|_{L^2(S)}^2 \right)$$

We give estimates for η .

Proposition 3.7. *We have the following estimates for η .*

$$(3.15) \quad \forall s \geq 0, \quad \|\nabla \eta(t)\|_{H^s(S)} \leq C_s |h(t)|_{s+\frac{1}{2}}$$

$$(3.16) \quad \forall s \in \mathbb{N}, \quad \|\nabla \partial_t \eta(t)\|_{H^s(S)} \leq C_s (1 + \|v\|_{L^\infty} + |\nabla h|_{L^\infty}) \left(\|v\|_{E^{s+1}} + |\nabla h|_{s+\frac{1}{2}} \right)$$

$$(3.17) \quad \forall s \in \mathbb{N}, \quad \|\eta\|_{W^{s,\infty}} \leq C_s |h|_{s,\infty}$$

$$(3.18) \quad \forall s \in \mathbb{N}, \quad \|\partial_t \eta\|_{W^{s,\infty}} \leq C_s (1 + |h|_{s,\infty}) \|v\|_{s,\infty}$$

Proposition 3.8. *For every $m \in \mathbb{N}$, we have*

$$(3.19) \quad \left\| \frac{f}{\partial_z \varphi} \right\|_m \leq \Lambda \left(\frac{1}{c_0}, |h|_{1,\infty} + \|f\|_{L^\infty} \right) \left(|h|_{m+\frac{1}{2}} + \|f\|_m \right)$$

Regularity of $\sqrt{\varepsilon} h$.

Proposition 3.9. *For every $m \in \mathbb{N}$, $\varepsilon \in (0, 1)$, we have the estimate*

$$(3.20) \quad \varepsilon |h(t)|_{m+\frac{1}{2}}^2 \leq \varepsilon |h_0|_{m+\frac{1}{2}}^2 + \varepsilon \int_0^t |v^b|_{m+\frac{1}{2}}^2 + \int_0^t \Lambda (|\nabla h|_{L^\infty(\mathbb{R}^2)} + \|v\|_{1,\infty}) \left(\|v\|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) d\tau$$

4. HIGH ORDER EQUATIONS

In this section we apply Z^α (let $|\alpha| = m$) to (1.4),(1.5) and rewrite the high order system in terms of

$$(\mathcal{V}^\alpha, \mathcal{Q}^\alpha, \mathcal{B}^\alpha) \equiv (Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta, Z^\alpha q - \partial_z^\varphi q Z^\alpha \eta, Z^\alpha B - \partial_z^\varphi B Z^\alpha \eta)$$

using linearization results. We will use similar notations as in [1], for convenience.

4.1. A Commutator estimate. For $i = 1, 2$ we write,

$$Z^\alpha \partial_i^\varphi f = \partial_i^\varphi Z^\alpha f - \partial_z^\varphi f \partial_i^\varphi Z^\alpha \eta + \mathcal{C}_i^\alpha(f)$$

$$\mathcal{C}_i^\alpha(f) \equiv \mathcal{C}_{i,1}^\alpha(f) + \mathcal{C}_{i,2}^\alpha(f) + \mathcal{C}_{i,3}^\alpha(f)$$

where

$$\begin{cases} \mathcal{C}_{i,1}^\alpha = -[Z^\alpha, \frac{\partial_i \varphi}{\partial_z \varphi}, \partial_z f] \\ \mathcal{C}_{i,2}^\alpha = -\partial_z f [Z^\alpha, \partial_i \varphi, \frac{1}{\partial_z \varphi}] - \partial_i \varphi \left(Z^\alpha \left(\frac{1}{\partial_z \varphi} \right) + \frac{Z^\alpha \partial_z \eta}{(\partial_z \varphi)^2} \right) \partial_z f \\ \mathcal{C}_{i,3}^\alpha = -\frac{\partial_i \varphi}{\partial_z \varphi} [Z^\alpha, \partial_z] f + \frac{\partial_i \varphi}{(\partial_z \varphi)^2} \partial_z f [Z^\alpha, \partial_z] \eta \end{cases}$$

For $i = 3$, result is very similar and we are suffice to replace $\partial_i \varphi$ be 1 in above terms. We need to estimate commutators.

Lemma 4.1. *For $1 \leq |\alpha| \leq m$, $i = 1, 2, 3$, we have*

$$\|\mathcal{C}_i^\alpha(f)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla f\|_{1,\infty} \right) \left(\|\nabla f\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

Proof. See Lemma 5.1 in [1] □

4.2. Divergence free condition for (v, B) .

$$Z^\alpha(\nabla^\varphi \cdot v) = 0$$

$$\nabla^\varphi \cdot (Z^\alpha v) - \partial_z^\varphi v \cdot \nabla^\varphi (Z^\alpha \varphi) + \sum_{i=1}^3 \mathcal{C}_i^\alpha(v_i) = 0$$

$$\nabla^\varphi \cdot (Z^\alpha v - \partial_z^\varphi v Z^\alpha \varphi) - (\nabla^\varphi \cdot \partial_z^\varphi v) Z^\alpha \varphi + \sum_{i=1}^3 \mathcal{C}_i^\alpha(v_i) = 0$$

second term is zero since ∂_i^φ is commute and $\nabla^\varphi \cdot v = 0$ Hence we get

$$(4.1) \quad \nabla^\varphi \cdot \mathcal{V}^\alpha + \mathcal{C}^\alpha(d_v) = 0, \quad \mathcal{C}^\alpha(d_v) \equiv \sum_{i=1}^3 \mathcal{C}_i^\alpha(v_i)$$

$$(4.2) \quad \nabla^\varphi \cdot \mathcal{B}^\alpha + \mathcal{C}^\alpha(d_B) = 0, \quad \mathcal{C}^\alpha(d_B) \equiv \sum_{i=1}^3 \mathcal{C}_i^\alpha(B_i)$$

where commutators $\mathcal{C}^\alpha(d_v), \mathcal{C}^\alpha(d_B)$ satisfy the following estimates by Lemma 4.1.

$$(4.3) \quad \|\mathcal{C}^\alpha(d_v)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla v\|_{1,\infty} \right) \left(\|\nabla v\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

$$(4.4) \quad \|\mathcal{C}^\alpha(d_B)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla B\|_{1,\infty} \right) \left(\|\nabla B\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

4.3. Navier-Stokes Equation with Lorentz force. We can directly use many useful results from [1].

$$Z^\alpha \{ \partial_z^\varphi v + (v \cdot \nabla^\varphi) v + \nabla^\varphi q - 2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi v) - (B \cdot \nabla^\varphi) B \} = 0$$

Transport

$$(4.5) \quad Z^\alpha (\partial_t^\varphi + (v \cdot \nabla^\varphi)) v = Z^\alpha (\partial_t + v_y \cdot \nabla_y v + V_z \partial_z) v$$

where $V_z = \frac{1}{\partial_z \varphi} (v \cdot \mathbf{N}^\varphi - \partial_t \eta)$ and where $\mathbf{N}^\varphi = (-\nabla_y \eta, 1)$. Then,

$$\begin{aligned} &= (\partial_t + v_y \cdot \nabla_y + V_z \partial_z) Z^\alpha v + (v \cdot Z^\alpha \mathbf{N}^\varphi - \partial_t Z^\alpha \eta) \partial_z^\varphi v - \partial_z^\varphi Z^\alpha \eta (v \cdot \mathbf{N}^\varphi - \partial_t \eta) \partial_z^\varphi v + \mathcal{C}^\alpha(\mathcal{T}_v) \\ &= (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^\alpha v) - \partial_z^\varphi v (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^\alpha \varphi) + \mathcal{C}^\alpha(\mathcal{T}_v) \end{aligned}$$

where $\mathcal{C}^\alpha(\mathcal{T}_v)$ is defined as

$$\begin{aligned} \mathcal{C}^\alpha(\mathcal{T}_v) &\equiv \sum_{i=1}^6 \mathcal{T}_i^\alpha \\ \mathcal{T}_1^\alpha &= [Z^\alpha, v_y, \partial_y v], \quad \mathcal{T}_2^\alpha = [Z^\alpha, V_z, \partial_z v], \quad \mathcal{T}_3^\alpha = \frac{1}{\partial_z \varphi} [Z^\alpha, v_z] \partial_z v, \\ \mathcal{T}_4^\alpha &= \left(Z^\alpha \left(\frac{1}{\partial_z \varphi} \right) + \frac{\partial_z Z^\alpha \eta}{(\partial_z \varphi)^2} \right) v_z \partial_z v, \quad \mathcal{T}_5^\alpha = v_z \partial_z v \frac{[Z^\alpha, \partial_z] \eta}{(\partial_z \varphi)^2} + V_z [Z^\alpha, \partial_z] v, \\ \mathcal{T}_6^\alpha &= [Z^\alpha, v_z, \frac{1}{\partial_z \varphi}] \partial_z v \end{aligned}$$

Estimate for $\mathcal{C}^\alpha(\mathcal{T}_v)$ is given in [1], (5.26).

$$(4.6) \quad \|\mathcal{C}^\alpha(\mathcal{T}_v)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{E^{2,\infty}} \right) \left(\|v\|_{E^m} + |h|_{m-\frac{1}{2}} \right)$$

Pressure

$$(4.7) \quad Z^\alpha \nabla^\varphi q = \nabla^\varphi (Z^\alpha q) - \partial_z^\varphi q \nabla^\varphi (Z^\alpha \varphi) + \mathcal{C}^\alpha(q)$$

with estimate

$$(4.8) \quad \|\mathcal{C}^\alpha(q)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla q\|_{1,\infty} \right) \left(\|\nabla q\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

Diffusion

$$(4.9) \quad \begin{aligned} Z^\alpha (-2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi v)) &= -2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi Z^\alpha v) + 2\varepsilon \nabla^\varphi \cdot (\partial_z^\varphi v \otimes \nabla^\varphi Z^\alpha \varphi + \nabla^\varphi Z^\alpha \varphi \otimes \partial_z^\varphi v) \\ &\quad + 2\varepsilon \partial_z^\varphi (\mathbf{S}^\varphi v) \nabla^\varphi (Z^\alpha \varphi) - \varepsilon \mathcal{D}^\alpha (\mathbf{S}^\varphi v) - \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha v) \end{aligned}$$

where $\mathcal{D}^\alpha (\mathbf{S}^\varphi v)$ and $(\mathcal{E}^\alpha v)$ are defined as

$$(\mathcal{E}^\alpha v)_{ij} \equiv \mathcal{C}_i^\alpha(v_j) + \mathcal{C}_j^\alpha(v_i), \quad \mathcal{D}^\alpha (\mathbf{S}^\varphi v)_i = 2\mathcal{C}_j^\alpha (\mathbf{S}^\varphi v)_{ij}$$

with estimate for $\mathcal{E}^\alpha(v)$

$$(4.10) \quad \|\mathcal{E}^\alpha(v)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla v\|_{1,\infty} \right) \left(\|v\|_m + \|\partial_z v\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

Lorentz force

$$\begin{aligned} (4.11) \quad -Z^\alpha (B \cdot \nabla^\varphi B) &= -Z^\alpha \left(\sum_{i=1}^3 B_i \partial_i^\varphi B \right) = -\sum_{i=1}^3 (Z^\alpha B_i \partial_i^\varphi B + B_i Z^\alpha \partial_i^\varphi B + [Z^\alpha, B_i, \partial_i^\varphi B]) \\ &= -(Z^\alpha B) \cdot \nabla^\varphi B - \sum_{i=1}^3 B_i (\partial_i^\varphi (Z^\alpha B) - \partial_z^\varphi B \partial_i^\varphi (Z^\alpha \varphi) + \mathcal{C}_i^\alpha(B)) - \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i^\varphi B] \\ &= -(Z^\alpha B) \cdot \nabla^\varphi B - (B \cdot \nabla^\varphi) (Z^\alpha B) + (\partial_z^\varphi B) (B \cdot \nabla^\varphi (Z^\alpha \varphi)) - \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha(B) - \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i^\varphi B] \end{aligned}$$

Let

$$\begin{aligned}\mathcal{C}^\alpha(\mathcal{T}_B) &\equiv \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i^\varphi B] \\ &= \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i B] - \sum_{i=1}^3 [Z^\alpha, B_i, \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z B]\end{aligned}$$

Then using Lemma 4.1, we have an estimate for $\mathcal{C}^\alpha(\mathcal{T}_B)$.

$$(4.12) \quad \|\mathcal{C}^\alpha(\mathcal{T}_B)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla B\|_{1,\infty} \right) \left(\|\nabla B\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

Now putting (4.5), (4.7), (4.9), (4.11) together, and using linearization of \mathcal{N} ,

$$\begin{aligned}0 &= D\mathcal{N}(v, B, q, \varphi) \cdot (Z^\alpha v, Z^\alpha B, Z^\alpha q, Z^\alpha \varphi) - (Z^\alpha v \cdot \nabla^\varphi) v \\ &\quad + \mathcal{C}^\alpha(\mathcal{T}_v) + \mathcal{C}^\alpha(q) - \varepsilon \mathcal{D}^\alpha(\mathbf{S}^\varphi v) - \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha v) - \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha(B) - \mathcal{C}^\alpha(\mathcal{T}_B)\end{aligned}$$

By proposition 2.4, (for $D\mathcal{N}(v, B, q, \varphi) \cdot (Z^\alpha v, Z^\alpha B, Z^\alpha q, Z^\alpha \varphi)$), we get the following.

$$\begin{aligned}(4.13) \quad &(\partial_t^\varphi + v \cdot \nabla^\varphi - 2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi \cdot)) \mathcal{V}^\alpha + \nabla^\varphi \mathcal{Q}^\alpha - (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \\ &= (Z^\alpha B \cdot \nabla^\varphi) B + Z^\alpha \varphi ((\partial_z^\varphi v \cdot \nabla^\varphi) v - (\partial_z^\varphi B \cdot \nabla^\varphi) B) + \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha(B) + \mathcal{C}^\alpha(\mathcal{T}_B) \\ &\quad + \varepsilon \mathcal{D}^\alpha(\mathbf{S}^\varphi v) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha v) - \mathcal{C}^\alpha(\mathcal{T}_v) - \mathcal{C}^\alpha(q)\end{aligned}$$

4.4. Faraday law.

$$Z^\alpha (\partial_t^\varphi B + (v \cdot \nabla^\varphi) B - (B \cdot \nabla^\varphi) v - 2\lambda \nabla^\varphi \cdot (\mathbf{S}^\varphi B)) = 0$$

Transport

$$Z^\alpha (\partial_t^\varphi + v \cdot \nabla^\varphi) B = Z^\alpha (\partial_t + v_y \cdot \nabla_y + V_z \partial_z) B$$

where

$$V_z = \frac{v \cdot \mathbf{N}^\varphi - \partial_t \varphi}{\partial_z \varphi}$$

Now we expand

$$\begin{aligned}&= \partial_t (Z^\alpha B) + \sum_{i=1}^2 (Z^\varphi v_i \partial_i B + v_i \partial_i Z^\alpha B + [Z^\alpha, v_i, \partial_i B]) + Z^\alpha V_z \cdot \partial_z B + V_z Z^\alpha \partial_z B + [Z^\alpha, V_z, \partial_z B] \\ &= (\partial_t + v_y \cdot \nabla_y) (Z^\alpha B) + Z^\alpha v_y \cdot \nabla_y B + \sum_{i=1}^2 [Z^\alpha, v_i, \partial_i B] + Z^\alpha \left(\frac{v \cdot \mathbf{N}^\varphi - \partial_t \varphi}{\partial_z \varphi} \right) \cdot \partial_z B + V_z \partial_z Z^\alpha B + V_z [Z^\alpha, \partial_z] B + [Z^\alpha, V_z, \partial_z B] \\ &= (\partial_t^\varphi + v_y \cdot \nabla_y + V_z \partial_z) (Z^\alpha B) + Z^\alpha \left(\frac{v \cdot \mathbf{N}^\varphi - \partial_t \varphi}{\partial_z \varphi} \right) \cdot \partial_z B + \mathcal{R}_1\end{aligned}$$

where

$$\mathcal{R}_1 \equiv V_z [Z^\alpha, \partial_z] B + Z^\alpha v_y \cdot \nabla_y B + \sum_{i=1}^2 [Z^\alpha, v_i, \partial_i B] + [Z^\alpha, V_z, \partial_z B]$$

So,

$$\begin{aligned}&= (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^\alpha B) + \partial_z B \left(\frac{1}{\partial_z \varphi} v \cdot Z^\alpha \mathbf{N}^\varphi + Z^\alpha \left(\frac{1}{\partial_z \varphi} \right) v \cdot \mathbf{N}^\varphi - \frac{1}{\partial_z \varphi} \partial_t Z^\alpha \varphi - \partial_t \varphi Z^\alpha \left(\frac{1}{\partial_z \varphi} \right) \right) \\ &\quad \partial_z B \left(\frac{1}{\partial_z \varphi} Z^\alpha v \cdot \mathbf{N}^\varphi + \sum_{i=1}^3 [Z^\alpha, \frac{1}{\partial_z \varphi}, v_i, \mathbf{N}_i^\varphi] - [Z^\alpha, \partial_t \varphi, \frac{1}{\partial_z \varphi}] \right) + \mathcal{R}_1\end{aligned}$$

Note that

$$Z^\alpha \left(\frac{1}{\partial_z \varphi} \right) = Z^{\alpha-1} \left(-\frac{Z \partial_z \varphi}{(\partial_z \varphi)^2} \right)$$

$$= -\frac{1}{(\partial_z \varphi)^2} (\partial_z Z^\alpha \varphi + [Z^\alpha, \partial_z] \varphi) - Z \partial_z \varphi Z^{\alpha-1} \left(\frac{1}{\partial_z \varphi} \right)^2 - [Z^{\alpha-1}, \frac{1}{(\partial_z \varphi)^2}, Z \partial_z \varphi]$$

Hence, transport part becomes

$$\begin{aligned} Z^\alpha (\partial_t^\varphi + v \cdot \nabla^\varphi) B &= (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^\alpha B) + (v \cdot Z^\alpha \mathbf{N}^\varphi - \partial_t Z^\alpha \varphi) \partial_z^\varphi B + \partial_z B Z^\alpha \left(\frac{1}{\partial_z \varphi} \right) (v \cdot \mathbf{N}^\varphi - \partial_t \varphi) \\ &\quad + \partial_z B \left(\frac{1}{\partial_z \varphi} Z^\alpha v \cdot \mathbf{N}^\varphi + \sum_{i=1}^3 [Z^\alpha, \frac{1}{\partial_z \varphi}, v_i, \mathbf{N}_i^\varphi] - [Z^\alpha, \partial_t \varphi, \frac{1}{\partial_z \varphi}] \right) + \mathcal{R}_1 \\ &= (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^\alpha B) + (v \cdot Z^\alpha \mathbf{N} - \partial_t Z^\alpha \varphi) \partial_z^\varphi B \\ &\quad + \partial_z B (v \cdot \mathbf{N}^\varphi - \partial_t \varphi) \left(-\frac{1}{(\partial_z \varphi)^2} (\partial_z Z^\alpha \varphi + [Z^\alpha, \partial_z] \varphi) - Z \partial_z \varphi Z^{\alpha-1} \left(\frac{1}{\partial_z \varphi} \right)^2 - [Z^{\alpha-1}, \frac{1}{(\partial_z \varphi)^2}, Z \partial_z \varphi] \right) + \tilde{\mathcal{R}}_1 \end{aligned}$$

where

$$\tilde{\mathcal{R}}_1 \equiv \partial_z B \left(\frac{1}{\partial_z \varphi} Z^\alpha v \cdot \mathbf{N}^\varphi + \sum_{i=1}^3 [Z^\alpha, \frac{1}{\partial_z \varphi}, v_i, \mathbf{N}_i^\varphi] - [Z^\alpha, \partial_t \varphi, \frac{1}{\partial_z \varphi}] \right) + \mathcal{R}_1$$

So,

$$= (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^\alpha B) + (v \cdot Z^\alpha \mathbf{N}^\varphi - \partial_t Z^\alpha \varphi) \partial_z^\varphi B - \partial_z^\varphi Z^\alpha \varphi (v \cdot \mathbf{N}^\varphi - \partial_t \varphi) \partial_z^\varphi B + \mathcal{C}^\alpha(\mathcal{T}_F)$$

At result,

$$(4.14) \quad Z^\alpha (\partial_t^\varphi + v \cdot \nabla^\varphi) B = (\partial_t^\varphi + v \cdot \nabla^\varphi) Z^\alpha B - \partial_z^\varphi B (\partial_t^\varphi + v \cdot \nabla^\varphi) Z^\alpha \varphi + \mathcal{C}^\alpha(\mathcal{T}_F)$$

where

$$\begin{aligned} \mathcal{C}^\alpha(\mathcal{T}_F) &= \partial_z B (v \cdot \mathbf{N}^\varphi - \partial_t \varphi) \left(-\frac{[Z^\alpha, \partial_z] \varphi}{(\partial_z \varphi)^2} - Z \partial_z \varphi Z^{\alpha-1} \left(\frac{1}{\partial_z \varphi} \right)^2 - [Z^{\alpha-1}, \frac{1}{(\partial_z \varphi)^2}, Z \partial_z \varphi] \right) \\ &\quad + \partial_z B \left(\frac{1}{\partial_z \varphi} Z^\alpha v \cdot \mathbf{N}^\varphi + \sum_{i=1}^3 [Z^\alpha, \frac{1}{\partial_z \varphi}, v_i, \mathbf{N}_i^\varphi] - [Z^\alpha, \partial_t \varphi, \frac{1}{\partial_z \varphi}] \right) \\ &\quad + V_z [Z^\alpha, \partial_z] B + Z^\alpha v_y \cdot \nabla_y B + \sum_{i=1}^2 [Z^\alpha, v_i, \partial_i B] + [Z^\alpha, V_z, \partial_z B] \end{aligned}$$

and using propositions in section 4 and Lemma 4.1 we get,

$$(4.15) \quad \|\mathcal{C}^\alpha(\mathcal{T}_F)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|v\|_{1,\infty} + \|B\|_{E^{2,\infty}} \right) \left(|h|_{m-\frac{1}{2}} + \|B\|_{E^m} + \|v\|_m \right)$$

$(\mathbf{B} \cdot \nabla^\varphi) \mathbf{v}$

$$\begin{aligned} (4.16) \quad -Z^\alpha (\mathbf{B} \cdot \nabla^\varphi v) &= -Z^\alpha \left(\sum_{i=1}^3 B_i \partial_i^\varphi v \right) = -\sum_{i=1}^3 (Z^\alpha B_i \partial_i^\varphi v + B_i Z^\alpha \partial_i^\varphi v + [Z^\alpha, B_i, \partial_i^\varphi v]) \\ &= -(Z^\alpha B) \cdot \nabla^\varphi v - \sum_{i=1}^3 B_i (\partial_i^\varphi (Z^\alpha v) - \partial_z^\varphi v \partial_i^\varphi (Z^\alpha \varphi) + \mathcal{C}_i^\alpha(v)) - \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i^\varphi v] \\ &= -(Z^\alpha B) \cdot \nabla^\varphi v - (B \cdot \nabla^\varphi) (Z^\alpha v) + (\partial_z^\varphi v) (B \cdot \nabla^\varphi (Z^\alpha \varphi)) - \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha(v) - \mathcal{C}^\alpha(\mathcal{T}_I) \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}^\alpha(\mathcal{T}_I) &\equiv \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i^\varphi v] \\ &= \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i v] - \sum_{i=1}^3 [Z^\alpha, B_i, \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z v] \end{aligned}$$

Then using Lemma 4.1, we have an estimate for $\mathcal{C}^\alpha(\mathcal{T}_I)$.

$$(4.17) \quad \|\mathcal{C}^\alpha(\mathcal{T}_I)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla v\|_{1,\infty} + \|\nabla B\|_{1,\infty} \right) \left(\|\nabla v\|_{m-1} + \|\nabla B\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

Diffusion

Diffusion part is just same as Navier-Stokes part.

$$(4.18) \quad \begin{aligned} Z^\alpha (-2\lambda \nabla^\varphi \cdot (\mathbf{S}^\varphi B)) &= -2\lambda \nabla^\varphi \cdot (\mathbf{S}^\varphi Z^\alpha B) + 2\lambda \nabla^\varphi \cdot (\partial_z^\varphi B \otimes \nabla^\varphi Z^\alpha \varphi + \nabla^\varphi Z^\alpha \varphi \otimes \partial_z^\varphi B) \\ &\quad + 2\lambda \partial_z^\varphi (\mathbf{S}^\varphi B) \nabla^\varphi (Z^\alpha \varphi) - \varepsilon \mathcal{D}^\alpha (\mathbf{S}^\varphi B) - \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha B) \end{aligned}$$

where definition and estimate of $\mathcal{D}^\alpha (\mathbf{S}^\varphi B)$ and $(\mathcal{E}^\alpha B)$ are same as before.

$$(\mathcal{E}^\alpha B)_{ij} \equiv \mathcal{C}_i^\alpha (B_j) + \mathcal{C}_j^\alpha (B_i), \quad \mathcal{D}^\alpha (\mathbf{S}^\varphi B)_i = 2\mathcal{C}_j^\alpha (\mathbf{S}^\varphi B)_{ij}$$

with estimate for $\mathcal{E}^\alpha (B)$

$$(4.19) \quad \|\mathcal{E}^\alpha (B)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla B\|_{1,\infty} \right) \left(\|v\|_m + \|\partial_z B\|_{m-1} + |h|_{m-\frac{1}{2}} \right)$$

Now, considering (4.14), (4.16), (4.18), we get

$$\begin{aligned} &D\mathcal{F}(v, B, \varphi) \cdot (Z^\alpha v, Z^\alpha B, Z^\alpha \varphi) - (Z^\alpha v \cdot \nabla^\varphi) B \\ &= -\mathcal{C}^\alpha (\mathcal{T}_F) + \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha (v) + \mathcal{C}^\alpha (\mathcal{T}_I) + \lambda \mathcal{D}^\alpha (\mathbf{S}^\varphi B) + \lambda \nabla^\varphi \cdot (\mathcal{E}^\alpha B) \end{aligned}$$

Using linearization,

$$(4.20) \quad \begin{aligned} &(\partial_t^\varphi + v \cdot \nabla^\varphi - 2\lambda \nabla^\varphi \cdot (\mathbf{S}^\varphi \cdot)) \mathcal{B}^\alpha - (B \cdot \nabla^\varphi) \mathcal{V}^\alpha \\ &= (Z^\alpha B \cdot \nabla^\varphi) v + Z^\alpha \varphi ((\partial_z^\varphi v \cdot \nabla^\varphi) B + (\partial_z^\varphi B \cdot \nabla^\varphi) v) - \mathcal{C}^\alpha (\mathcal{T}_F) \\ &\quad + \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha (v) + \mathcal{C}^\alpha (\mathcal{T}_I) + \lambda \mathcal{D}^\alpha (\mathbf{S}^\varphi B) + \lambda \nabla^\varphi \cdot (\mathcal{E}^\alpha B) \end{aligned}$$

4.5. Kinetic Boundary. For the boundary condition $Z^\alpha = D^\alpha$ in fact since $\alpha_3 = 0$. This is just same as Lemma 5.7 in [1],

$$(4.21) \quad \partial_z Z^\alpha h - v^b \cdot Z^\alpha \mathbf{N} - \mathcal{V}^\alpha \cdot \mathbf{N} = \mathcal{C}^\alpha (h)$$

where

$$\mathcal{C}^\alpha (h) \equiv -[Z^\alpha, v_y^b, \nabla_y h] - \frac{(\partial_z v)^b}{\partial_z \varphi} \cdot \mathbf{N} Z^\alpha h$$

with estimate for $\mathcal{C}^\alpha (h)$

$$(4.22) \quad |\mathcal{C}^\alpha (h)|_{L^2} \leq \Lambda \left(\frac{1}{c_0}, \|v\|_{E^{1,\infty}} + |h|_{2,\infty} \right) (\|v\|_{E^m} + |h|_m)$$

4.6. Continuity of Stress-Tensor. This is also same as Lemma 5.6 in [1],

$$Z^\alpha ((q - gh)\mathbf{N} - 2\varepsilon \mathbf{S}^\varphi v \mathbf{N}) = 0$$

similarly as [1],

$$(4.23) \quad 2\varepsilon \mathbf{S}^\varphi \mathcal{V}^\alpha \mathbf{N} - (Z^\alpha q - g Z^\alpha h) \mathbf{N} + (2\varepsilon \mathbf{S}^\varphi v - (q - gh)) Z^\alpha \mathbf{N} = \mathcal{C}^\alpha (\partial) - 2\varepsilon Z^\alpha h \partial_z^\varphi (\mathbf{S}^\varphi v) \mathbf{N}$$

where

$$\mathcal{C}^\alpha (\partial) \equiv -\varepsilon \mathcal{E}^\alpha (v) - \sum_{\substack{\beta+\gamma=\alpha, \\ 0<|\beta|<|\alpha|}} \varepsilon Z^\beta (\mathbf{S}^\varphi v) Z^\gamma \mathbf{N} + \sum_{\substack{\beta+\gamma=\alpha, \\ 0<|\beta|<|\alpha|}} \varepsilon Z^\beta (q - gh) Z^\gamma \mathbf{N}$$

with estimate for $\mathcal{C}^\alpha (\partial)$

$$(4.24) \quad |\mathcal{C}^\alpha (\partial)|_{L^2} \leq \varepsilon \Lambda \left(\frac{1}{c_0}, \|v\|_{E^{2,\infty}} + |h|_{2,\infty} \right) (|v^b|_m + |h|_m)$$

5. PRESSURE ESTIMATES

In this section we estimate the pressure for any smooth solution (v, B, q, h) . We divide q into $q = q^E + q^{NS}$, where

$$\begin{aligned}\Delta^\varphi q^E &= -\nabla^\varphi \cdot (v \cdot \nabla^\varphi v) + \nabla^\varphi \cdot (B \cdot \nabla^\varphi B), \quad q^E|_{z=0} = gh \\ \Delta^\varphi q^{NS} &= 0, \quad q^{NS}|_{z=0} = 2\varepsilon \mathbf{S}^\varphi v \mathbf{n} \cdot \mathbf{n}\end{aligned}$$

We express Δ^φ as elliptic operator. (See [1] section 6)

$$\Delta^\varphi f = \frac{1}{\partial_z \varphi} \nabla \cdot (E \nabla f)$$

where

$$E = \begin{pmatrix} \partial_z \varphi & 0 & -\partial_1 \varphi \\ 0 & \partial_z \varphi & -\partial_2 \varphi \\ -\partial_1 \varphi & -\partial_2 \varphi & \frac{1+(\partial_1 \varphi)^2+(\partial_2 \varphi)^2}{\partial_z \varphi} \end{pmatrix} = \frac{1}{\partial_z \varphi} P P^*$$

where

$$P = \begin{pmatrix} \partial_z \varphi & 0 & 0 \\ 0 & \partial_z \varphi & 0 \\ -\partial_1 \varphi & -\partial_2 \varphi & 1 \end{pmatrix}$$

E is positive symmetric matrix and there exists $\delta(c_0) > 0$ such that

$$EX \cdot X \geq \delta |X|^2, \quad \forall X \in \mathbb{R}^3$$

if $\|\nabla_y \varphi\|_{L^\infty} \leq \frac{1}{c_0}$, and $\partial_z \varphi \leq c_0 > 0$. We have an estimate

$$(5.1) \quad \|E\|_{W^{k,\infty}} \leq \Lambda\left(\frac{1}{c_0}, |h|_{k+1,\infty}\right)$$

Also, using the following decomposition,

$$E = \mathbf{I}_A + \tilde{E}, \quad \tilde{E} = \begin{pmatrix} \partial_z \eta & 0 & -\partial_1 \eta \\ 0 & \partial_z \eta & -\partial_2 \eta \\ -\partial_1 \eta & -\partial_2 \eta & \frac{A((\partial_1 \eta)^2+(\partial_2 \eta)^2)-\partial_z \eta}{A\partial_z \eta} \end{pmatrix}, \quad \mathbf{I}_A = \text{diag}(A, A, 1/A)$$

we also get an estimate.

$$(5.2) \quad \|\tilde{E}\|_{H^s} \leq \Lambda\left(\frac{1}{c_0}, |h|_{1,\infty}\right) |h|_{s+\frac{1}{2}}$$

We employ the following lemmas about elliptic problem, from [1]. First Lemma is for q^E .

Lemma 5.1. *For elliptic equation in S ,*

$$-\nabla \cdot (E \nabla \rho) = \nabla \cdot F, \quad \rho|_{z=0} = 0$$

we have the estimates :

$$(5.3) \quad \|\nabla \rho\| \leq \Lambda\left(\frac{1}{c_0}, |h|_{1,\infty}\right) \|F\|_{L^2}, \quad \|\nabla^2 \rho\| \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty}\right) (\|\nabla \cdot F\| + \|F\|_1)$$

$$(5.4) \quad \|\nabla \rho\|_k \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + \|F\|_{H_{tan}^2} + \|\nabla \cdot F\|_{H_{tan}^1}\right) (|h|_{k+\frac{1}{2}} + \|F\|_k), \quad k \geq 1$$

$$(5.5) \quad \|\partial_{zz} \rho\|_{k-1} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + \|F\|_{H_{tan}^2} + \|\nabla \cdot F\|_{H_{tan}^1}\right) (|h|_{k+\frac{1}{2}} + \|F\|_k + \|\nabla \cdot F\|_{k-1}), \quad k \geq 2$$

Proof. See Lemma 6.1 in [1]. □

Secone Lemma is for q^{NS} .

Lemma 5.2. *For elliptic equation in S ,*

$$-\nabla \cdot (E \nabla \rho) = 0, \quad \rho|_{z=0} = f^b$$

we have the estimates :

$$(5.6) \quad \|\nabla \rho\|_{H^k} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |f^b|_{1,\infty} + |f^b|_{5/2}\right) (|h|_{k+1/2} + |f^b|_{k+1/2})$$

Proof. See Lemma 6.2 in [1]. □

Using above two lemmas, we can get estimates for q^E, q^{NS} .

Proposition 5.3. *For q^E , we have the estimates :*

$$(5.7) \quad \|\nabla q^E\|_{m-1} + \|\partial_{zz} q^E\|_{m-2} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + \|v\|_{E^{1,\infty}} + \|B\|_{E^{1,\infty}} + \|v\|_{E^3} + \|B\|_{E^3}\right) \\ \times (\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}})$$

$$(5.8) \quad \|\nabla q^E\|_{1,\infty} + \|\partial_{zz} q^E\|_{L^\infty} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_4 + \|v\|_{E^{1,\infty}} + \|B\|_{E^{1,\infty}} + \|v\|_{E^4} + \|B\|_{E^4}\right)$$

$$(5.9) \quad \|\nabla q^E\|_{2,\infty} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_5 + \|v\|_{E^{1,\infty}} + \|B\|_{E^{1,\infty}} + \|v\|_{E^5} + \|B\|_{E^5}\right)$$

Proof. See Proposition 6.4 in [1]. We just suffice to add similar terms about B . □

Proposition 5.4. *For q^{NS} , we have the estimates for $m \leq 1$:*

$$(5.10) \quad \|\nabla q^{NS}\|_{H^{m-1}} \leq \varepsilon \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_4 + \|v\|_{E^{2,\infty}} + \|v\|_{E^4}\right) (|v^b|_{m+\frac{1}{2}} + |h|_{m+\frac{1}{2}})$$

$$(5.11) \quad \|\nabla q^{NS}\|_{L^\infty} \leq \varepsilon \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_4 + \|v\|_{E^{2,\infty}} + \|v\|_{E^4}\right)$$

Remark 5.5. Note that q^{NS} can be estimated in standard sobolev space, not conormal.

Proof. See Proposition 6.3 in [1]. □

The following proposition will be used for Taylor sign condition.

Proposition 5.6. *For $T \in (0, T^\varepsilon)$, we have the estimate.*

$$(5.12) \quad \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} \leq \int_0^T \Lambda\left(\frac{1}{c_0}, |h|_6 + |h|_{3,\infty} + \|v\|_6 + \|B\|_6 + \|\partial_z v\|_4 + \|\partial_z B\|_4 + \|v\|_{E^{2,\infty}} + \|B\|_{E^{2,\infty}}\right) \\ \cdot (1 + \varepsilon \|\partial_{zz} v\|_{L^\infty} + \lambda \|\partial_{zz} B\|_{L^\infty} + \varepsilon \|\partial_{zz} v\|_3 + \lambda \|\partial_{zz} B\|_3) d\tau$$

Proof. This is revised version of proposition 6.5 in [1]. It's also very similar to [1]. First,

$$\triangle^\varphi q^E = -\nabla^\varphi \cdot (v \cdot \nabla^\varphi v) + \nabla^\varphi \cdot (B \cdot \nabla^\varphi B), \quad q^E|_{z=0} = gh$$

Take ∂_t then

$$\nabla \cdot (E \nabla \partial_t q^E) = -\nabla \cdot (\partial_t (P(v \cdot \nabla^\varphi v))) + \nabla \cdot (\partial_t (P(B \cdot \nabla^\varphi B))) - \nabla \cdot (\partial_t E \nabla P^E), \quad \partial_t q^E|_{z=0} = g \partial_t h$$

As like in [1], divide this as $q^E = q^i + q^B$ so that

$$\nabla \cdot (E \nabla \partial_t P^i) = -\nabla \cdot (\partial_t (P(v \cdot \nabla^\varphi v))) + \nabla \cdot (\partial_t (P(B \cdot \nabla^\varphi B))) - \nabla \cdot (\partial_t E \nabla P^E), \quad \partial_t P^i|_{z=0} = 0$$

$$\nabla \cdot (E \nabla \partial_t P^B) = 0, \quad \partial_t P^B|_{z=0} = g \partial_t h$$

Estimate for q^B is exactly same as [1], and for q^i , we use Lemma 6.6 in [1] where F also includes similar structure for B . We get

$$\int_0^T |(\partial_z \partial_t q^i)^b|_{L^\infty} \leq \int_0^T \Lambda\left(\frac{1}{c_0}, \|v\|_{E^5} + \|v\|_{E^{2,\infty}} + \|v\|_6 + \|B\|_{E^5} + \|B\|_{E^{2,\infty}} + \|B\|_6 + |h|_6 + |h|_{3,\infty}\right) \\ \cdot (1 + \|\partial_t v\|_{L^\infty} + \|\partial_t v\|_3 + \|\partial_t B\|_{L^\infty} + \|\partial_t B\|_3) ds$$

Estimate for $\|\partial_z v\|_3 + \|\partial_t v\|_{L^\infty}$ is given in [1]. And, B part come from problem setting, i.e, similar to [1], (no pressure and v, B are entangled each other)

$$\|\partial_z B\|_3 + \|\partial_t B\|_{L^\infty} \leq \Lambda\left(\frac{1}{c_0}, \|B\|_{E^5} + \|B\|_{E^{2,\infty}} + \|v\|_{E^5} + \|v\|_{E^{2,\infty}} + |h|_5 + |h|_{2,\infty}\right) \\ \cdot (1 + \lambda \|\partial_{zz} B\|_{L^\infty} + \lambda \|\partial_{zz} B\|_3)$$

q^B is already given in [1].

$$\|\nabla q^B\|_{L^\infty} \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |\partial_t h|_3 \right)$$

Putting altogether, we get just 'B-added' version proposition.

$$(5.13) \quad \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty} \leq \int_0^T \Lambda \left(\frac{1}{c_0}, \|v\|_6 + \|\partial_z v\|_4 + \|v\|_{E^{2,\infty}} + \|B\|_6 + \|\partial_z B\|_4 + \|B\|_{E^{2,\infty}} + |h|_6 + |h|_{3,\infty} \right) \\ \cdot (1 + \varepsilon \|\partial_{zz} v\|_{L^\infty} + \varepsilon \|\partial_{zz} v\|_3 + \lambda \|\partial_{zz} B\|_{L^\infty} + \lambda \|\partial_{zz} B\|_3) ds$$

□

6. L^2 ENERGY ESTIMATE

Proposition 6.1. *For any smooth solution of (1.4), (1.5), we have the following zero-order energy estimate.*

$$\frac{d}{dt} \left(\int_S |v|^2 dV_t + g \int_{\partial S} |h|^2 dy + \int_S |B|^2 dV_t \right) + 4\varepsilon \int_S |\mathbf{S}^\varphi v|^2 dV_t + 4\lambda \int_S |\mathbf{S}^\varphi B|^2 dV_t = 0$$

Proof. Multiplying v and integrating on S for Navier-Stokes, and also using boundary condition, we get

$$\frac{d}{dt} \int_S |v|^2 dV_t + 4\varepsilon \int_S |\mathbf{S}^\varphi v|^2 dV_t - \int_{\partial S} |v|^2 (h_t - v^b \cdot \mathbf{N}) dy = 2 \int_{\partial S} (2\varepsilon \mathbf{S}^\varphi v - q\mathbf{I}) \mathbf{N} \cdot v dy + 2 \int_S v \cdot (B \cdot \nabla^\varphi) B dV_t$$

Then using kinematic boundary condition and Continuity of stress tensor condition,

$$\frac{d}{dt} \int_S |v|^2 dV_t + 4\varepsilon \int_S |\mathbf{S}^\varphi v|^2 dV_t = 2 \int_S v \cdot (B \cdot \nabla^\varphi) B dV_t - 2 \int_{\partial S} gh(\mathbf{N} \cdot v) dy \\ (6.1) \quad \frac{d}{dt} \left(\int_S |v|^2 dV_t + g \int_{\partial S} |h|^2 dy \right) + 4\varepsilon \int_S |\mathbf{S}^\varphi v|^2 dV_t = 2 \int_S v \cdot (B \cdot \nabla^\varphi) B dV_t$$

Multiplying B and integrating on S for Faraday's Law, we get

$$\frac{d}{dt} \int_S |B|^2 dV_t + 4\lambda \int_S |\mathbf{S}^\varphi B|^2 dV_t = 2 \int_S B \cdot (B \cdot \nabla^\varphi) v dV_t + 4\lambda \int_{\partial S} \mathbf{S}^\varphi B (\mathbf{N} \cdot B) dV_t$$

Using divergence free condition, we know that

$$(6.2) \quad \int_S B \cdot (B \cdot \nabla^\varphi) v dV_t = \int_{\partial S} (B \cdot v)(B \cdot \mathbf{n}) dy - \int_S v \cdot (B \cdot \nabla^\varphi) B dV_t \\ = - \int_S v \cdot (B \cdot \nabla^\varphi) B dV_t$$

So, Faraday's Law part become,

$$(6.3) \quad \frac{d}{dt} \int_S |B|^2 dV_t + 4\lambda \int_S |\mathbf{S}^\varphi B|^2 dV_t = -2 \int_S v \cdot (B \cdot \nabla^\varphi) B dV_t$$

Adding (6.1) to (6.3), we get the result. □

7. HIGHER ORDER ENERGY ESTIMATE

Using the results of section 3,4,5, we make high order energy estimate.

7.1. Navier-Stokes Equation with Lorentz force. We apply L^2 energy estimate in section 6 to (4.1), (4.13), (4.21), (4.23), to get

$$(7.1) \quad \begin{aligned} & \frac{d}{dt} \int_S |\mathcal{V}^\alpha|^2 dV_t + 4\varepsilon \int_S |\mathbf{S}^\varphi \mathcal{V}^\alpha|^2 dV_t - 2 \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t \\ &= \mathcal{R}_S + \mathcal{R}_C + \mathcal{R}_P + 2 \int_{z=0} (2\varepsilon \mathbf{S}^\varphi \mathcal{V}^\alpha - \mathcal{Q}^\alpha Id) \mathbf{n} \cdot \mathcal{V}^\alpha dy \end{aligned}$$

where \mathcal{R}_S and \mathcal{R}_C are defined,

$$(7.2) \quad \mathcal{R}_S = 2 \int_S (\varepsilon \mathbf{D}^\alpha(\mathbf{S}^\varphi v) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(v))) \cdot \mathcal{V}^\alpha dV_t$$

$$(7.3) \quad \mathcal{R}_C = -2 \int_S ((\mathcal{C}^\alpha(\mathcal{T}_v) + \mathcal{C}^\alpha(q)) \cdot \mathcal{V}^\alpha - \mathcal{C}^\alpha(d_v) \mathcal{Q}^\alpha) dV_t$$

and \mathcal{R}_P is defined as

$$(7.4) \quad \mathcal{R}_P = 2 \int_S \left\{ (Z^\alpha B \cdot \nabla^\varphi) B + Z^\alpha \varphi ((\partial_z^\varphi v \cdot \nabla^\varphi) v + (\partial_z^\varphi B \cdot \nabla^\varphi) B) + \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha(B) + \mathcal{C}^\alpha(\mathcal{T}_B) \right\} \cdot \mathcal{V}^\alpha dV_t$$

7.2. Faraday law. Similarly we can get the following energy estimate using (4.2), (4.20).

$$(7.5) \quad \begin{aligned} & \frac{d}{dt} \int_S |\mathcal{B}^\alpha|^2 dV_t + 4\lambda \int_S |\mathbf{S}^\varphi \mathcal{B}^\alpha|^2 dV_t + 2 \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t \\ &= 4\lambda \int_{z=0} (\mathbf{S}^\varphi \mathcal{B}^\alpha) \mathbf{n} \cdot \mathcal{B}^\alpha dy + 2 \int_S \{ (Z^\alpha B \cdot \nabla^\varphi) v + Z^\alpha \varphi ((\partial_z^\varphi v \cdot \nabla^\varphi) B + (\partial_z^\varphi B \cdot \nabla^\varphi) v) \\ & \quad - \mathcal{C}^\alpha(\mathcal{T}_F) + \sum_{i=1}^3 v_i \mathcal{C}_i^\alpha(B) + \mathcal{C}^\alpha(\mathcal{T}_I) + \varepsilon \mathbf{D}^\alpha(\mathbf{S}^\varphi B) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(B)) \} \cdot \mathcal{B}^\alpha dV_t \\ &= 4\lambda \int_{z=0} (\mathbf{S}^\varphi \mathcal{B}^\alpha) \mathbf{n} \cdot \mathcal{B}^\alpha dy + \mathcal{R}_{S_B} + \mathcal{R}_{C_B} + \mathcal{R}_{P_B} = \mathcal{R}_{S_B} + \mathcal{R}_{C_B} + \mathcal{R}_{P_B} \end{aligned}$$

where

$$\mathcal{R}_{S_B} = 2 \int_S \{ \varepsilon \mathbf{D}^\alpha(\mathbf{S}^\varphi B) + \varepsilon \nabla^\varphi \cdot (\mathcal{E}^\alpha(B)) \} \cdot \mathcal{B}^\alpha dV_t$$

$$\mathcal{R}_{C_B} = -2 \int_S \mathcal{C}^\alpha(\mathcal{T}_F) \cdot \mathcal{B}^\alpha dV_t$$

$$\mathcal{R}_{P_B} = 2 \int_S \left\{ (Z^\alpha B \cdot \nabla^\varphi) v + Z^\alpha \varphi ((\partial_z^\varphi v \cdot \nabla^\varphi) B + (\partial_z^\varphi B \cdot \nabla^\varphi) v) + \sum_{i=1}^3 v_i \mathcal{C}_i^\alpha(B) + \mathcal{C}^\alpha(\mathcal{T}_I) \right\} \cdot \mathcal{B}^\alpha dV_t$$

Remark 7.1.

$$\left(Z^\alpha B - \frac{\partial_z B}{\partial_z \varphi} Z^\alpha \varphi \right)_{z=0} = 0$$

since

1) when $\alpha_3 \neq 0$, trivially this is zero by $\frac{z}{1-z}$.

2) when $\alpha_3 = 0$, $\left(D_{x,y}^\alpha B - \frac{\partial_z B}{\partial_z \varphi} D_{x,y}^\alpha \varphi \right)_{z=0}$. First term is trivially zero and the second term is zero via $\partial_z B = 0$ as B is identically zero throughout vacuum and whole B should be smooth.

7.3. More commutator estimates. We need to know the estimates of $\mathcal{R}_S, \mathcal{R}_C, \mathcal{R}_P, \mathcal{R}_{S_B}, \mathcal{R}_{C_B}, \mathcal{R}_{P_B}$. Note that the estimates of \mathcal{R}_S is given in [1] as following.

1) Estimate of \mathcal{R}_S

$$(7.6) \quad \begin{aligned} \|\mathcal{R}_S\| &\leq \varepsilon \Lambda_\infty \{ \|\nabla \mathcal{V}^\alpha\| (\|\mathcal{V}^\alpha\| + \|\partial_z v\|_{m-1} + |h|_{m+\frac{1}{2}}) \\ &\quad + (\|v\|_{E^m}^2 + |h|_{m+\frac{1}{2}}^2) + \|\partial_{zz} v\|_{L^\infty} (|h|_m^2 + \|\mathcal{V}^\alpha\|^2) \} \end{aligned}$$

2) Estimate of \mathcal{R}_C For about \mathcal{R}_C , difference to [1] comes from the estimate of $\mathcal{C}^\alpha(q^E)$. We just add some terms about B , so

$$\|\mathcal{C}^\alpha(q^E)\| \|\mathcal{V}^m\| \leq \Lambda_\infty (\|v\|_{E^m} + \|B\|_{E^m} + |h|_m) \|\mathcal{V}^m\|$$

So we get

$$\|\mathcal{R}_C\| \leq \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_m + \varepsilon |h|_{m+\frac{1}{2}} + \varepsilon |v^b|_{m+\frac{1}{2}} \right) \|\mathcal{V}^m\|$$

So, similarly as [1],

$$(7.7) \quad \|\mathcal{R}_C\| \leq \Lambda_\infty \left(\varepsilon \|\nabla \mathcal{V}^m\| \|\mathcal{V}^m\| + \|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\mathcal{V}^m\|^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right)$$

Note that we don't have B^b since v^b come from q^{NS} which has nothing to do with B . For about \mathcal{R}_S , it is exactly same as in [1],

3) Estimate of \mathcal{R}_P

$$\mathcal{R}_P = 2 \int_S \left\{ (Z^\alpha B \cdot \nabla^\varphi) B + Z^\alpha \varphi ((\partial_z^\varphi v \cdot \nabla^\varphi) v + (\partial_z^\varphi B \cdot \nabla^\varphi) B) + \sum_{i=1}^3 B_i \mathcal{C}_i^\alpha(B) + \mathcal{C}^\alpha(\mathcal{T}_B) \right\} \cdot \mathcal{V}^\alpha dV_t$$

Basically we divide this as just L^2 . And we know that

$$\|\mathcal{C}_i^\alpha(B)\| \leq \Lambda \left(\frac{1}{c_0}, |h|_{2,\infty} + \|\nabla B\|_{1,\infty} \right) (\|\nabla B\|_{m-1} + |h|_{m-\frac{1}{2}})$$

from [1] Lemma 5.1. For $\mathcal{C}^\alpha(\mathcal{T}_B)$,

$$\begin{aligned} \mathcal{C}^\alpha(\mathcal{T}_B) &= \sum_{i=1}^3 [Z^\alpha, B_i, \partial_i^\varphi B] \leq \Lambda \left(\frac{1}{c_0}, \|B\|_{1,\infty} + \|\partial_i^\varphi B\|_{1,\infty} \right) (\|B\|_{m-1} + \|\partial_i^\varphi B\|_{m-1}) \\ &\leq \Lambda \left(\frac{1}{c_0}, \|B\|_{E^{2,\infty}} + |h|_{2,\infty} \right) (\|B\|_{E^m} + |h|_m) \end{aligned}$$

Hence we get

$$(7.8) \quad \|\mathcal{R}_P\| \leq \Lambda \left(\frac{1}{c_0}, \|v\|_{E^{1,\infty}} + \|B\|_{E^{2,\infty}} + |h|_{2,\infty} \right) (\|\mathcal{V}^m\| + \|B\|_{E^m} + |h|_m)$$

4) Estimate of \mathcal{R}_{S_B} is nearly same as [1]. We just change v of (7.6) into B .

$$(7.9) \quad \begin{aligned} \|\mathcal{R}_{S_B}\| &\leq \varepsilon \Lambda_\infty \{ \|\nabla \mathcal{B}^\alpha\| (\|\mathcal{B}^\alpha\| + \|\partial_z B\|_{m-1} + |h|_{m+\frac{1}{2}}) \\ &\quad + (\|B\|_{E^m}^2 + |h|_{m+\frac{1}{2}}^2) + \|\partial_{zz} B\|_{L^\infty} (|h|_m^2 + \|\mathcal{B}^\alpha\|^2) \} \end{aligned}$$

5) Estimate of \mathcal{R}_{C_B} is simply,

$$(7.10) \quad \|\mathcal{R}_{C_B}\| \leq \Lambda_\infty (\|B\|_m + \|\partial_z B\|_{m-1} + |h|_m) \|\mathcal{B}^\alpha\|$$

6) Estimate of \mathcal{R}_{P_B} can be estimated similarly as \mathcal{R}_P

$$(7.11) \quad \|\mathcal{R}_{P_B}\| \leq \Lambda \left(\frac{1}{c_0}, \|v\|_{E^{1,\infty}} + \|B\|_{E^{2,\infty}} + |h|_{2,\infty} \right) (\|\mathcal{B}^m\| + \|B\|_{E^m} + |h|_m)$$

7.4. Energy estimate for Navier-Stokes Equation with Lorentz Force. From now on, we define,

$$\Lambda_\infty(t) \doteq \Lambda \left(\frac{1}{c_0}, \|v(t)\|_{E^{2,\infty}} + \sqrt{\varepsilon} \|\partial_{zz}v(t)\|_{L^\infty} + \|B(t)\|_{E^{2,\infty}} + \sqrt{\lambda} \|\partial_{zz}B(t)\|_{L^\infty} + \|v(t)\|_{E^4} + |h|_4 \right)$$

Using (7.1),

$$(7.12) \quad \frac{d}{dt} \frac{1}{2} \left(\int_S |\mathcal{V}^\alpha|^2 dV_t + \int_{\partial S} (g - \partial_z^\varphi q^E) |Z^\alpha h|^2 dy \right) + 4\varepsilon \int_S |\mathbf{S}^\varphi \mathcal{V}^\alpha|^2 dV_t - 2 \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t \\ = \mathcal{R}_S + \mathcal{R}_C + \mathcal{R}_P + \tilde{\mathcal{R}}_B$$

where estimate of $\tilde{\mathcal{R}}_B$ is given by (See [1])

$$(7.13) \quad \|\tilde{\mathcal{R}}_B\| \leq \varepsilon \Lambda_\infty((1 + \|\partial_{zz}v\|_{L^\infty})|h|_m + |v^b|_m)(\mathcal{V}^\alpha)^b|_{L^2}$$

For this estimate, the only difference from [1] comes from q^E . In [1], q^E is not estimated, so $(g - \partial_z^\varphi q^E)$ remains. So, estimate of $\tilde{\mathcal{R}}_B$ is exactly same as in [1]. Now under assumption of

$$(7.14) \quad \partial_z \varphi \geq c_0, \quad |h|_{2,\infty} \leq \frac{1}{c_0}, \quad g - (\partial_z^\varphi q^E)|_{z=0} \geq \frac{c_0}{2}, \quad \forall t \in [0, T^\varepsilon]$$

We get

$$\|\mathcal{V}^m(t)\|^2 + |h(t)|_m^2 + 4\varepsilon \int_S \|\mathbf{S}^\varphi \mathcal{V}^m\|^2 dV_t - 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t ds \\ \leq \Lambda_0 (\|\mathcal{V}^m(0)\|^2 + |h(0)|_m^2) + \int_0^t \varepsilon \Lambda_\infty \|\nabla \mathcal{V}^m\| \left(\|\mathcal{V}^m\| + \|v\|_{E^m} + |h|_m + |h|_{m+\frac{1}{2}} \right) dV_t \\ \int_0^t \Lambda_\infty (1 + |(\partial_z \partial_t q^E)^b|_{L^\infty}) \left(\|\mathcal{V}^m\|^2 + \|v\|_{E^m}^2 + \|B\|_{E^m}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) dV_t$$

where

$$\|\mathcal{V}^m(t)\|^2 = \sum_{|\alpha| \leq m} \|\mathcal{V}^\alpha(t)\|^2, \quad \|\mathbf{S}^\varphi \mathcal{V}^m(t)\|^2 = \sum_{|\alpha| \leq m} \|\mathbf{S}^\varphi \mathcal{V}^\alpha(t)\|^2$$

Using Young's inequality, adapted Korn's inequality (proposition 3.6) and proposition 3.9 (for regularity of $\sqrt{\varepsilon}h$), we get the following proposition.

Proposition 7.2. *Under the assumption of*

$$\partial_z \varphi \geq c_0, \quad |h|_{2,\infty} \leq \frac{1}{c_0}, \quad g - (\partial_z^\varphi q^E)|_{z=0} \geq \frac{c_0}{2}, \quad \forall t \in [0, T^\varepsilon]$$

we have the following estimate.

$$(7.15) \quad \|\mathcal{V}^m(t)\|^2 + |h(t)|_m^2 + \varepsilon |h(t)|_{m+\frac{1}{2}}^2 + \varepsilon \int_S \|\nabla \mathcal{V}^m\|^2 dV_t - 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t ds \\ \leq \Lambda_0 \left(\|\mathcal{V}^m(0)\|^2 + |h(0)|_m^2 + \varepsilon |h(0)|_{m+\frac{1}{2}}^2 \right) + \int_0^t \Lambda_\infty (1 + |(\partial_z \partial_t q^E)^b|_{L^\infty}) \left(\|\mathcal{V}^m\|^2 + \|B\|_m^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) dV_t \\ + \int_0^t \Lambda_\infty (\|\partial_z v\|_{m-1}^2 + \|\partial_z B\|_{m-1}^2) dV_t$$

($\|v\|_{E^m}$ is absorbed into $\|\mathcal{V}^m\|$ and $\|\partial_z v\|_{m-1}$. Also for B .)

7.5. Energy estimate for Faraday Law. Using (7.5),

$$\begin{aligned} \frac{d}{dt} \int_S |\mathcal{B}^\alpha|^2 dV_t + 4\lambda \int_S |\mathbf{S}^\varphi \mathcal{B}^\alpha|^2 dV_t + 2 \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t \\ = \mathcal{R}_{S_B} + \mathcal{R}_{C_B} + \mathcal{R}_{P_B} \end{aligned}$$

Using estimate for $\mathcal{R}_{S_B}, \mathcal{R}_{C_B}, \mathcal{R}_{P_B}$,

$$\begin{aligned} \|\mathcal{B}^m\|^2 + 4\lambda \int_0^t \int_S |\mathbf{S}^\varphi \mathcal{B}^\alpha|^2 dV_t ds + 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t ds \leq \Lambda_0 (\|\mathcal{B}^m\|^2) \\ + \int_0^t \varepsilon \Lambda_\infty \|\nabla \mathcal{B}^m\| \left(\|\mathcal{B}^m\| + \|B\|_{E^m} + |h|_m + |h|_{m+\frac{1}{2}} \right) dV_t \\ + \int_0^t \Lambda_\infty \left(\|\mathcal{B}^m\|^2 + \|v\|_m^2 + \|B\|_{E^m}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) dV_t \end{aligned}$$

There is no $\partial_z \partial_t q^E$ part here since it come from $\tilde{\mathcal{R}}_B$ which come from (7.9). Again, using Young's inequality, proposition 3.6 and proposition 3.9, we get

Proposition 7.3. *We have the following estimate.*

$$\begin{aligned} (7.16) \quad \|\mathcal{B}^m\|^2 + \lambda \int_0^t \int_S |\nabla \mathcal{B}^\alpha|^2 dV_t ds + 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \mathcal{B}^\alpha \cdot \mathcal{V}^\alpha dV_t ds \\ \leq \Lambda_0 (\|\mathcal{B}^m(0)\|^2) + \int_0^t \Lambda_\infty \left(\|\mathcal{B}^m\|^2 + \|v\|_m^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) dV_t + \int_0^t \Lambda_\infty (\|\partial_z B\|_{m-1}^2) dV_t \end{aligned}$$

We sum two main estimates (7.15) and (7.16), then we see that we should estimate some terms in Λ_∞ (such as $\|\partial_z v\|_{k,\infty}$) and $\|\partial_z v\|_{m-1}^2 + \|\partial_z B\|_{m-1}^2$. Note that we should use proposition 5.6 to estimate $|(\partial_z \partial_t q^E)^b|_{L^\infty}$ on the right hand side.

8. NORMAL ESTIMATE

From the results of section 7, we should estimate $\|\partial_z v\|_{m-1}$ and $\|\partial_z B\|_{m-1}$, because they are not controlled by $\|v\|_m, \|B\|_m$. First, the following Lemma 8.1 is true for both v and B .

Lemma 8.1. *For every integer $m \geq 1$, normal part of $\partial_z v, \partial_z B$ can be estimated as follow.*

$$\begin{aligned} \|\partial_z v \cdot \mathbf{n}\|_{m-1} &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) \left(\|\mathcal{V}^m\| + |h|_{m-\frac{1}{2}} \right) \\ \|\partial_z B \cdot \mathbf{n}\|_{m-1} &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla B\|_{L^\infty} \right) \left(\|\mathcal{B}^m\| + |h|_{m-\frac{1}{2}} \right) \end{aligned}$$

Proof. See [1] □

The following Lemma 8.2 is also valid for B , since Lemma 8.2 uses only definition, normal component.(which comes from divergence free).

Lemma 8.2. *For every integer $k \geq 0$, when we define,*

$$S_n^v = \Pi \mathbf{S}^\varphi v \mathbf{n}, \quad S_n^B = \Pi \mathbf{S}^\varphi B \mathbf{n}, \quad \Pi = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$$

Then we get

$$\begin{aligned} (8.1) \quad \|\partial_z v\|_k &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{L^\infty} \right) \left(\|S_n^v\|_k + |h|_{k+\frac{1}{2}} + \|v\|_{k+1} \right) \\ \|\partial_{zz} v\|_k &\leq \Lambda \left(\frac{1}{c_0}, \|v\|_{E^{2,\infty}} \right) \left(\|\nabla S_n^v\|_k + |h|_{k+\frac{3}{2}} + \|v\|_{k+2} \right) \end{aligned}$$

Exactly sams for B , since these come from definition and divergence free in fact.

$$\begin{aligned} (8.2) \quad \|\partial_z B\|_k &\leq \Lambda \left(\frac{1}{c_0}, \|\nabla B\|_{L^\infty} \right) \left(\|S_n^B\|_k + |h|_{k+\frac{1}{2}} + \|B\|_{k+1} \right) \\ \|\partial_{zz} B\|_k &\leq \Lambda \left(\frac{1}{c_0}, \|B\|_{E^{2,\infty}} \right) \left(\|\nabla S_n^B\|_k + |h|_{k+\frac{3}{2}} + \|B\|_{k+2} \right) \end{aligned}$$

Proof. See [1] □

Now we proceed something similar to [1] proposition 8.3. Similarly, $m - 2$ will be optimal regularity for S_n^v , S_n^B , because of regularity of pressure.

Proposition 8.3. *We have the following estimate.*

$$(8.3) \quad \begin{aligned} & \|S_n^v\|_{m-2}^2 + 2\varepsilon \int_0^T \|\nabla^\varphi S_n^v\|_{m-2}^2 ds - 2 \int_0^T \int_S Z^\alpha S_n^v \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^B dV_t ds \\ & \leq \Lambda_0 \|S_n^v(0)\|_{m-2}^2 + \int_0^T \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) ds \\ & \quad + 2\varepsilon \int_0^T |v^b|_{m+\frac{1}{2}} \|S_n^v\|_{m-2} ds + \Lambda_0 \varepsilon \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds \end{aligned}$$

Proof. Apply ∇^φ to equation to get

$$\partial_t^\varphi \nabla^\varphi v + (v \cdot \nabla^\varphi) \nabla^\varphi v + (\nabla^\varphi v)^2 - (B \cdot \nabla^\varphi) \nabla^\varphi B - (\nabla^\varphi B)^2 + (D^\varphi)^2 q - \varepsilon \Delta^\varphi \nabla^\varphi v = 0$$

Take symmetric and then get (\mathbf{D}^φ is symmetric)

$$\begin{aligned} & \partial_t^\varphi \mathbf{S}^\varphi v + (v \cdot \nabla^\varphi) \mathbf{S}^\varphi v + \frac{1}{2} ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^T)^2) \\ & - (B \cdot \nabla^\varphi) \mathbf{S}^\varphi B - \frac{1}{2} ((\nabla^\varphi B)^2 + ((\nabla^\varphi B)^T)^2) + (D^\varphi)^2 q - \varepsilon \Delta^\varphi (\mathbf{S}^\varphi v) = 0 \end{aligned}$$

Applying $\cdot \mathbf{n}$ and tangential projection operator Π , we get

$$(8.4) \quad \partial_t^\varphi S_n^v + (v \cdot \nabla^\varphi) S_n^v - (B \cdot \nabla^\varphi) S_n^B - \varepsilon \Delta^\varphi (S_n^v) = F_S$$

where $F_S = F_S^1 + F_S^2 + F_S^3$,

$$\begin{aligned} F_S^1 &= -\frac{1}{2} \Pi ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^T)^2) \mathbf{n} + (\partial_t \Pi + v \cdot \nabla^\varphi \Pi) \mathbf{S}^\varphi v \mathbf{n} + \Pi \mathbf{S}^\varphi v (\partial_t \mathbf{n} + v \cdot \nabla^\varphi \mathbf{n}) \\ F_S^2 &= -2\varepsilon \partial_i^\varphi \Pi \partial_i^\varphi (\mathbf{S}^\varphi v \mathbf{n}) - 2\varepsilon \Pi (\partial_i^\varphi (\mathbf{S}^\varphi v) \partial_i^\varphi \mathbf{n}) - \varepsilon (\Delta^\varphi \Pi) \mathbf{S}^\varphi v \mathbf{n} - \varepsilon \Pi \mathbf{S}^\varphi v \Delta^\varphi \mathbf{n} - \Pi ((D^\varphi)^2 q) \mathbf{n} \\ F_S^3 &= \frac{1}{2} \Pi ((\nabla^\varphi B)^2 + ((\nabla^\varphi B)^T)^2) \mathbf{n} - (B \cdot \nabla^\varphi \Pi) \mathbf{S}^\varphi B \mathbf{n} - \Pi \mathbf{S}^\varphi B (B \cdot \nabla^\varphi \mathbf{n}) \end{aligned}$$

Note for example,

$$\begin{aligned} & (\partial_t^\varphi \mathbf{S}^\varphi v) \mathbf{n} = \partial_t^\varphi (\mathbf{S}^\varphi v \mathbf{n}) - \mathbf{S}^\varphi v \partial_t \mathbf{n} \\ & \Pi (\partial_t^\varphi \mathbf{S}^\varphi v) \mathbf{n} = \Pi (\partial_t^\varphi (\mathbf{S}^\varphi v \mathbf{n}) - \mathbf{S}^\varphi v \partial_t \mathbf{n}) \\ & = \Pi \partial_t^\varphi (\mathbf{S}^\varphi v \mathbf{n}) - \Pi (\mathbf{S}^\varphi v \partial_t \mathbf{n}) = \partial_t^\varphi (\Pi \mathbf{S}^\varphi v \mathbf{n}) - (\partial_t \Pi) \mathbf{S}^\varphi v \mathbf{n} - \Pi (\mathbf{S}^\varphi v \partial_t \mathbf{n}) \\ & = \partial_t^\varphi (S_n^v) - (\partial_t \Pi) \mathbf{S}^\varphi \mathbf{n} - \Pi \mathbf{S}^\varphi v \partial_t \mathbf{n} \end{aligned}$$

We can easily estimate F_S^1, F_S^2, F_S^3 .

$$(8.5) \quad \begin{aligned} \|F_S^1\|_{m-2} &\leq \Lambda_\infty \left(\|S_n^v\|_{m-2} + |h|_{m-\frac{1}{2}} + \|v\|_{m-1} \right) \\ \|F_S^2\|_{m-2} &\leq \Lambda_\infty \varepsilon \left(\|\nabla^\varphi S_n^v\|_{m-2} + |h|_{m+\frac{1}{2}} + |v^b|_{m+\frac{1}{2}} \right) + \Lambda_\infty \left(\|v\|_{E^m} + |h|_{m-\frac{1}{2}} \right) \\ \|F_S^3\|_{m-2} &\leq \Lambda_\infty \left(\|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} + \|B\|_{m-1} \right) \end{aligned}$$

Boundary compatibility condition,

$$S_n^v|_{z=0} = 0$$

Remark 8.4. In above three estimates, order of v and h gives critical optimal criteria. For h ,

$$\begin{aligned} F_S^1 &\sim \nabla \mathbf{n} \sim \nabla \nabla \varphi \sim |h|_{m-2+2-\frac{1}{2}} \sim |h|_{m-\frac{1}{2}} \\ F_S^2 &\sim \varepsilon \Delta \Pi \sim \varepsilon \Delta \nabla \varphi \sim \varepsilon |\varphi|_{m-2+3} \sim \varepsilon |h|_{m+\frac{1}{2}} \end{aligned}$$

Already we got full regularity about h , so cannot raise. F_S^3 is similar to F_S^1 . For v ,

$$\begin{aligned} F_S^1 &\sim \|v\|_{m-1}, \|S_n^v\|_{m-2} \\ F_S^2 &\sim \Lambda_\infty \|\nabla q^E\|_{E^{m-1}} \sim \|v\|_{E^m} \end{aligned}$$

v regularity in F_S^2 also maximal, although we cannot try $(m - 1)$ order already. F_S^3 is similar.

We know revised basic L^2 energy estimate for our equation.

$$(8.6) \quad \frac{1}{2} \frac{d}{dt} \int_S |S_n^v|^2 dV_t + \varepsilon \int_S |\nabla^\varphi S_n^v|^2 dV_t = \int_S F_S \cdot S_n^v dV_t + \int_S S_n^v \cdot (B \cdot \nabla^\varphi) S_n^B dV_t$$

Applying Z^α to (8.4),

$$\partial_t^\varphi Z^\alpha S_n^v + (v \cdot \nabla^\varphi) Z^\alpha S_n^v - (B \cdot \nabla^\varphi) Z^\alpha S_n^B - \varepsilon \Delta^\varphi Z^\alpha (S_n^v) = Z^\alpha (F_S) + \mathcal{C}_S$$

where

$$\mathcal{C}_S = \mathcal{C}_S^1 + \mathcal{C}_S^2 + \mathcal{C}_S^3$$

with

$$\mathcal{C}_S^1 = [Z^\alpha v_y] \cdot \nabla_y S_n^v + [Z^\alpha, V_z] \partial_z S_n^v \doteq \mathcal{C}_{S_y}^1 + \mathcal{C}_{S_z}^1$$

$$\mathcal{C}_S^2 = \varepsilon [Z^\alpha, \Delta^\varphi] S_n^v$$

$$\mathcal{C}_S^3 = -[Z^\alpha B_y] \cdot \nabla_y S_n^B + [Z^\alpha, \frac{B \cdot N}{\partial_z \varphi}] \partial_z S_n^B \doteq \mathcal{C}_{S_y}^3 + \mathcal{C}_{S_z}^3$$

High order estimate becomes,

$$\frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n^v|^2 dV_t + \varepsilon \int_S |Z^\alpha \nabla^\varphi S_n^v|^2 dV_t = \int_S (Z^\alpha (F_S) + \mathcal{C}_S) \cdot Z^\alpha S_n^v dV_t + \int_S Z^\alpha S_n^v \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^B dV_t$$

The last term will be canceled with similar term from faraday's Law. Estimates of $\mathcal{C}_S^1, \mathcal{C}_S^2$ are given in [1], using some variants of Hardy inequality.

$$\|\mathcal{C}_S^1\| \leq \Lambda_\infty (\|S_n^v\|_{m-2} + \|v\|_{E^{m-1}} + |h|_{m-\frac{1}{2}})$$

$$\left| \int_S \mathcal{C}_S^2 \cdot Z^\alpha S_n^v dV_t \right| \leq \Lambda_0 (\varepsilon^{1/2} \|\nabla Z^\alpha S_n^v\| + \|S_n\|_{m-2}) \left(\varepsilon^{1/2} \|\nabla S_n^v\|_{m-3} + \|S_n\|_{m-2} + \Lambda_\infty (|h|_{m-\frac{3}{2}} + \varepsilon^{1/2} |h|_{m-\frac{1}{2}}) \right)$$

The only difference to [1] is

$$\int_S Z^\alpha (F_S^3) \cdot Z^\alpha S_n^v dV_t, \quad \int_S \mathcal{C}_S^3 \cdot Z^\alpha S_n^v dV_t$$

Since structure of F_S^3, \mathcal{C}_S^3 are nearly similar to F_S^1, \mathcal{C}_S^1 , we get similar result, but replacing v into B . Hence when $\alpha = m-2$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n^v|^2 dV_t + \varepsilon \int_S |Z^\alpha \nabla^\varphi S_n^v|^2 dV_t \leq \int_S Z^\alpha S_n^v \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^B dV_t \\ & + \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) \\ & + \varepsilon |v^b|_{m+\frac{1}{2}} \|S_n^v\|_{m-2} + \Lambda_0 \varepsilon \|\nabla S_n^v\|_{m-3}^2 \end{aligned}$$

This implies,

$$\begin{aligned} & \|S_n^v\|_{m-2}^2 + 2\varepsilon \int_0^T \|\nabla^\varphi S_n^v\|_{m-2}^2 ds - 2 \int_0^T \int_S Z^\alpha S_n^v \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^B dV_t ds \\ & \leq \Lambda_0 \|S_n^v(0)\|_{m-2}^2 + \int_0^T \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) ds \\ & \quad + 2\varepsilon \int_0^T |v^b|_{m+\frac{1}{2}} \|S_n^v\|_{m-2} ds + \Lambda_0 \varepsilon \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds \end{aligned}$$

We first stop here and use induction later after putting together this with estimate from Faraday's law. \square

Now we do similar thing for Faraday law.

Proposition 8.5. *We have the estimate. (not induction yet.)*

$$\begin{aligned} (8.7) \quad & \|S_n^B\|_{m-2}^2 + 2\lambda \int_0^T \|\nabla^\varphi S_n^B\|_{m-2}^2 ds + 2 \int_0^T \int_S Z^\alpha S_n^v \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^B dV_t ds \\ & \leq \Lambda_0 \|S_n^B(0)\|_{m-2}^2 + \int_0^T \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\lambda} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) ds \\ & \quad + \Lambda_0 \lambda \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds \end{aligned}$$

Proof. Apply ∇^φ to equation to get

$$\partial_t^\varphi \nabla^\varphi B + (v \cdot \nabla^\varphi) \nabla^\varphi B + (\nabla^\varphi B)(\nabla^\varphi v) - (B \cdot \nabla^\varphi) \nabla^\varphi v - (\nabla^\varphi v)(\nabla^\varphi B) - \lambda \Delta^\varphi \nabla^\varphi B = 0$$

Take symmetric and then get (D^φ is symmetric)

$$\begin{aligned} & \partial_t^\varphi \mathbf{S}^\varphi B + (v \cdot \nabla^\varphi) \mathbf{S}^\varphi B - (B \cdot \nabla^\varphi) \mathbf{S}^\varphi v - \lambda \Delta^\varphi (\mathbf{S}^\varphi B) \\ & + \frac{1}{2} ((\nabla^\varphi B)(\nabla^\varphi v) - (\nabla^\varphi v)(\nabla^\varphi B) + (\nabla^\varphi v)^T (\nabla^\varphi B)^T - (\nabla^\varphi B)^T (\nabla^\varphi v)^T) = 0 \end{aligned}$$

Applying \mathbf{n} and Π , we get

$$\partial_t^\varphi S_n^B + (v \cdot \nabla^\varphi) S_n^B - (B \cdot \nabla^\varphi) S_n^v - \lambda \Delta^\varphi (S_n^B) = E_S$$

where $E_S = E_S^1 + E_S^2 + E_S^3$.

$$\begin{aligned} (8.8) \quad E_S^1 &= (\partial_t \Pi + v \cdot \nabla^\varphi \Pi) \mathbf{S}^\varphi B \mathbf{n} + \Pi \mathbf{S}^\varphi B (\partial_t \mathbf{n} + v \cdot \nabla^\varphi \mathbf{n}) \\ &- \frac{1}{2} ((\nabla^\varphi B)(\nabla^\varphi v) - (\nabla^\varphi v)(\nabla^\varphi B) + (\nabla^\varphi v)^T (\nabla^\varphi B)^T - (\nabla^\varphi B)^T (\nabla^\varphi v)^T) = 0 \\ E_S^2 &= -2\lambda \partial_i^\varphi \Pi \partial_i^\varphi (\mathbf{S}^\varphi B \mathbf{n}) - 2\lambda \Pi (\partial_i^\varphi (\mathbf{S}^\varphi B) \partial_i^\varphi \mathbf{n}) - \lambda (\Delta^\varphi \Pi) \mathbf{S}^\varphi B \mathbf{n} - \lambda \Pi \mathbf{S}^\varphi B \Delta^\varphi \mathbf{n} \\ E_S^3 &= -(B \cdot \nabla^\varphi \Pi) \mathbf{S}^\varphi v \mathbf{n} - \Pi \mathbf{S}^\varphi v (B \cdot \nabla^\varphi \mathbf{n}) \end{aligned}$$

We estimate E_S^1, E_S^2, E_S^3 . (we don't have pressure estimate here.)

$$\begin{aligned} (8.9) \quad \|E_S^1\|_{m-2} &\leq \Lambda_\infty \left(\|h\|_{m-\frac{1}{2}} + \|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + \|B\|_{m-1} + \|v\|_{m-2} \right) \\ \|E_S^2\|_{m-2} &\leq \Lambda_\infty \lambda \left(\|\nabla^\varphi S_n^B\|_{m-2} + \|h\|_{m+\frac{1}{2}} \right) \\ \|E_S^3\|_{m-2} &\leq \Lambda_\infty \left(\|h\|_{m-\frac{1}{2}} + \|S_n^v\|_{m-2} + \|B\|_{m-2} + \|v\|_{m-1} \right) \end{aligned}$$

Boundary condition,

$$S_n^B|_{z=0} = 0$$

We can get similar L^2 estimate as like this.

$$(8.10) \quad \frac{1}{2} \frac{d}{dt} \int_S |S_n^B|^2 dV_t + \lambda \int_S |\nabla^\varphi S_n^B|^2 dV_t = \int_S E_S \cdot S_n^v dV_t + \int_S S_n^B \cdot (B \cdot \nabla^\varphi) S_n^v dV_t$$

Applying Z^α , we get

$$\partial_t^\varphi Z^\alpha S_n^B + (v \cdot \nabla^\varphi) Z^\alpha S_n^B - (B \cdot \nabla^\varphi) Z^\alpha S_n^v - \lambda \Delta^\varphi Z^\alpha (S_n^B) = Z^\alpha (E_S) + \bar{\mathcal{C}}_S$$

where

$$\bar{\mathcal{C}}_S = \bar{\mathcal{C}}_S^1 + \bar{\mathcal{C}}_S^2 + \bar{\mathcal{C}}_S^3$$

with

$$\begin{aligned} \bar{\mathcal{C}}_S^1 &= [Z^\alpha v_y] \cdot \nabla_y S_n^B + [Z^\alpha, V_z] \partial_z S_n^B \doteq \bar{\mathcal{C}}_{S_y}^1 + \bar{\mathcal{C}}_{S_z}^1 \\ \bar{\mathcal{C}}_S^2 &= \lambda [Z^\alpha, \Delta^\varphi] S_n^B \\ \bar{\mathcal{C}}_S^3 &= -[Z^\alpha B_y] \cdot \nabla_y S_n^v + [Z^\alpha, \frac{B \cdot N}{\partial_z \varphi}] \partial_z S_n^v \doteq \bar{\mathcal{C}}_{S_y}^3 + \bar{\mathcal{C}}_{S_z}^3 \end{aligned}$$

Our high order estimate becomes,

$$\frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n^B|^2 dV_t + \lambda \int_S |Z^\alpha \nabla^\varphi S_n^B|^2 dV_t = \int_S (Z^\alpha (E_S) + \bar{\mathcal{C}}_S) \cdot Z^\alpha S_n^B dV_t + \int_S Z^\alpha S_n^B \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^v dV_t$$

The only differences are estimates of F_S, E_S and $\mathcal{C}_S, \bar{\mathcal{C}}_S$. Since estimates of these terms are similar, so when $\alpha = m-2$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n^B|^2 dV_t + \lambda \int_S |Z^\alpha \nabla^\varphi S_n^B|^2 dV_t \leq \int_S Z^\alpha S_n^B \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^v dV_t \\ & + \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + \|h\|_{m-\frac{1}{2}} + \sqrt{\lambda} \|h\|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + \|h\|_{m-\frac{1}{2}} \right) + \Lambda_0 \lambda \|\nabla S_n^B\|_{m-3}^2 \end{aligned}$$

(Comparing with previous proposition, $\lambda |v^b|_{m+\frac{1}{2}} \|S_n^B\|_{m-2}$ disappeared since it come from pressure estimate q^{NS} .)

$$\|S_n^B\|_{m-2}^2 + 2\lambda \int_0^T \|\nabla^\varphi S_n^B\|_{m-2}^2 ds - 2 \int_0^T \int_S Z^\alpha S_n^B \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^v dV_t ds$$

$$\begin{aligned} &\leq \Lambda_0 \|S_n^B(0)\|_{m-2}^2 + \int_0^T \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\lambda} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) ds \\ &\quad + \Lambda_0 \lambda \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds \end{aligned}$$

To make cancellation with previous proposition, we calculate,

$$\int_S f \cdot (B \cdot \nabla^\varphi) g dV_t = \int_{\partial S} f \cdot g (B \cdot \mathbf{n}) - \int_S \nabla^\varphi \cdot B (f \cdot g) - \int_S g \cdot (B \cdot \nabla^\varphi) f = - \int_S g \cdot (B \cdot \nabla^\varphi) f$$

So, finally

$$\begin{aligned} &\|S_n^B\|_{m-2}^2 + 2\lambda \int_0^T \|\nabla^\varphi S_n^B\|_{m-2}^2 ds + 2 \int_0^T \int_S Z^\alpha S_n^v \cdot (B \cdot \nabla^\varphi) Z^\alpha S_n^B dV_t ds \\ &\leq \Lambda_0 \|S_n^B(0)\|_{m-2}^2 + \int_0^T \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\lambda} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) ds \\ &\quad + \Lambda_0 \lambda \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds \end{aligned}$$

We stop here since we should use cancellation before use induction. \square

Now we put together above two propositions, cancel some terms, do something([1] last part of section 8 page 49) on $2\varepsilon \int_0^T |v^b|_{m+\frac{1}{2}} \|S_n^v\|_{m-2} ds$, and finally use induction to $\varepsilon \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds, \lambda \int_0^T \|\nabla S_n^B\|_{m-3}^2 ds$.

Proposition 8.6. *We have the estimate.*

$$\begin{aligned} (8.11) \quad &\|S_n^v\|_{m-2}^2 + \|S_n^B\|_{m-2}^2 + 2\varepsilon \int_0^T \|\nabla^\varphi S_n^v\|_{m-2}^2 ds + 2\lambda \int_0^T \|\nabla^\varphi S_n^B\|_{m-2}^2 ds \\ &\leq \Lambda_0 \left(\|S_n^v(0)\|_{m-2}^2 + \|S_n^B(0)\|_{m-2}^2 \right) \\ &\quad + \int_0^T \Lambda_\infty \left(\|\mathcal{V}^m\|^2 + \|\mathcal{B}^m\|^2 + \|S_n^v\|_{m-2}^2 + \|S_n^B\|_{m-2}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) \\ &\quad + \int_0^T \Lambda_\infty \left(\|\partial_z v\|_{m-1}^2 + \|\partial_z B\|_{m-1}^2 \right) + \varepsilon \int_0^T \|\nabla \mathcal{V}^m\|^2 \end{aligned}$$

Proof. We sum two proposition to get, (after cancel a term on left hand side),

$$\begin{aligned} &\|S_n^v\|_{m-2}^2 + \|S_n^B\|_{m-2}^2 + 2\varepsilon \int_0^T \|\nabla^\varphi S_n^v\|_{m-2}^2 ds + 2\lambda \int_0^T \|\nabla^\varphi S_n^B\|_{m-2}^2 ds \\ &\leq \Lambda_0 \left(\|S_n^v(0)\|_{m-2}^2 + \|S_n^B(0)\|_{m-2}^2 \right) \\ &\quad + \int_0^T \Lambda_\infty \left(\|v\|_{E^m} + \|B\|_{E^m} + |h|_{m-\frac{1}{2}} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right) \left(\|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_{m-\frac{1}{2}} \right) ds \\ &\quad + 2\varepsilon \int_0^T |v^b|_{m+\frac{1}{2}} \|S_n^v\|_{m-2} ds + \Lambda_0 \left(\varepsilon \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds + \lambda \int_0^T \|\nabla S_n^B\|_{m-3}^2 ds \right) \end{aligned}$$

Using trace estimate, we get

$$|v^b|_{m+\frac{1}{2}} \leq \|\nabla \mathcal{V}^m\| + \|\mathcal{V}\|_m + \Lambda_\infty |h|_{m+\frac{1}{2}}$$

then we get

$$\varepsilon \int_0^T |v^b|_{m+\frac{1}{2}} \|S_n^v\|_{m-2} ds \leq \varepsilon \int_0^T \|\nabla \mathcal{V}^m\|^2 + \int_0^T \Lambda_\infty \left(\|\mathcal{V}^m\|^2 + \|S_n^v\|_{m-2}^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right)$$

We can replace $\|\nabla^\varphi S_n^v\|_k$ by $\|\nabla S_n^v\|_k$, and use induction for $\left(\varepsilon \int_0^T \|\nabla S_n^v\|_{m-3}^2 ds + \lambda \int_0^T \|\nabla S_n^B\|_{m-3}^2 ds \right)$ \square

9. L^∞ TYPE ESTIMATE

From now on, we set $\varepsilon = \lambda$. In this section, we estimate all L^∞ type terms in Λ_∞ . Proposition 9.1 in [1] holds for v, B both, since they come from sobolev embedding or structure of Alinhac's unknown. We redefine modified energy form,

$$\begin{aligned} \mathcal{Q}_m(t) &\doteq |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 + \|\mathcal{V}^m\|^2 + \|\mathcal{B}^m\|^2 + \|S_n^v\|_{m-2}^2 + \|S_n^v\|_{1,\infty}^2 + \varepsilon \|\partial_z S_n^v\|_{L^\infty}^2 \\ &\quad + \|S_n^B\|_{m-2}^2 + \|S_n^B\|_{1,\infty}^2 + \varepsilon \|\partial_z S_n^B\|_{L^\infty}^2 \end{aligned}$$

From proposition 7.3 we know that we should estimate $\|\partial_z v\|_{m-1}, \|\partial_z B\|_{m-1}$. But, we have only $m-2$ order estimate in section 8. We will get $\|\partial_z v\|_{m-1}, \|\partial_z B\|_{m-1}$ at section 10. In this section, we control some L^∞ type terms here and $\|S_n^v\|_{m-2}, \|S_n^B\|_{m-2}$ would be sufficient to estimate them. First we state a corollary which resembles corollary 9.3 in [1]. These terms come from L^∞ type terms in Λ_∞ .

Corollary 9.1. *when $m \geq 6$, at each t ,*

$$(9.1) \quad \|v\|_{2,\infty} + \|\partial_z v\|_{1,\infty} + \sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty} + \|B\|_{2,\infty} + \|\partial_z B\|_{1,\infty} + \sqrt{\varepsilon} \|\partial_{zz} B\|_{L^\infty} + |h|_{4,\infty} \leq \Lambda \left(\frac{1}{c_0}, \mathcal{Q}_m \right)$$

Hence,

$$\begin{aligned} \Lambda_{m,\infty}(t) &\doteq \Lambda \left(\frac{1}{c_0}, \|v\|_m + \|\partial_z v\|_{m-2} + \|B\|_m + \|\partial_z B\|_{m-2} + |h|_m + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right. \\ &\quad \left. + \|v\|_{E^2,\infty} + \sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty} + \|B\|_{E^2,\infty} + \sqrt{\varepsilon} \|\partial_{zz} B\|_{L^\infty} \right) \\ &\leq \Lambda \left(\frac{1}{c_0}, \mathcal{Q}_m \right), \quad m \geq 6 \end{aligned}$$

Hence we control \mathcal{Q}_m instead of $\Lambda_{m,\infty}$. Now we start with estimate of $\|S_n^v\|_{1,\infty}$ and $\|S_n^B\|_{1,\infty}$.

9.1. zero-order estimate. First we note that zero-order estimate is just come from maximal principle of transport equation, estimate (9.12), but we should be careful that we have one more term which from Lorentz force. From (8.4),

$$\partial_t^\varphi S_n^v + (v \cdot \nabla^\varphi) S_n^v - (B \cdot \nabla^\varphi) S_n^B - \varepsilon \Delta^\varphi (S_n^v) = F_S$$

we get,

$$\|S_n^v(t)\|_{L^\infty} \leq \|S_n^v(0)\|_{L^\infty} + \int_0^t (\|F_S\|_{L^\infty} + \|(B \cdot \nabla) S_n^B\|_{L^\infty})$$

Main problem is that $\|\nabla S_n^B\|_{L^\infty}$ is not controlled by $\Lambda_{\infty,m}$. In fact we need $\|\partial_{zz} B\|_{L^\infty}$ to control this. This is same for Faraday law. From the equation of S_n^B ,

$$\partial_t^\varphi S_n^B + (v \cdot \nabla^\varphi) S_n^B - (B \cdot \nabla^\varphi) S_n^v - \varepsilon \Delta^\varphi (S_n^B) = E_S$$

we get

$$\|S_n^B(t)\|_{L^\infty} \leq \|S_n^B(0)\|_{L^\infty} + \int_0^t (\|E_S\|_{L^\infty} + \|(B \cdot \nabla) S_n^v\|_{L^\infty})$$

We also need $\|\partial_{zz} v\|_{L^\infty}$ to control $\|(B \cdot \nabla) S_n^v\|_{L^\infty}$. So instead of attacking S_n^v, S_n^B separately, we treat them at the same time. By adding and deducting two equations, we get,

$$(9.2) \quad \partial_t^\varphi (S_n^v + S_n^B) + ((v - B) \cdot \nabla^\varphi) (S_n^v + S_n^B) - \varepsilon \Delta^\varphi (S_n^v + S_n^B) = F_S + E_S$$

and

$$(9.3) \quad \partial_t^\varphi (S_n^v - S_n^B) + ((v + B) \cdot \nabla^\varphi) (S_n^v - S_n^B) - \varepsilon \Delta^\varphi (S_n^v - S_n^B) = F_S - E_S$$

Using maximal principle, from (9.2),

$$(9.4) \quad \|(S_n^v + S_n^B)(t)\|_{L^\infty} \leq \|(S_n^v + S_n^B)(0)\|_{L^\infty} + \int_0^t \|F_S + E_S\|_{L^\infty}$$

where

$$\int_0^t \|F_S\|_{L^\infty} \leq \varepsilon \int_0^t \|\mathcal{S}^\varphi \mathcal{V}^m\|^2 + (1 + \varepsilon) \int_0^t \Lambda_{\infty,m}, \quad \int_0^t \|E_S\|_{L^\infty} \leq \int_0^t \Lambda_{\infty,m}$$

This result is nearly same for (9.3).

$$(9.5) \quad \|(S_n^v + S_n^B)(t)\|_{L^\infty} \leq \|S_n^v(0)\|_{L^\infty} + \|S_n^B(0)\|_{L^\infty} + \varepsilon \int_0^t \|\mathbf{S}^\varphi \mathcal{V}^m\|^2 + (1 + \varepsilon) \int_0^t \Lambda_{\infty, m}$$

$$(9.6) \quad \|(S_n^v - S_n^B)(t)\|_{L^\infty} \leq \|S_n^v(0)\|_{L^\infty} + \|S_n^B(0)\|_{L^\infty} + \varepsilon \int_0^t \|\mathbf{S}^\varphi \mathcal{V}^m\|^2 + (1 + \varepsilon) \int_0^t \Lambda_{\infty, m}$$

By (9.5) and (9.6),

$$(9.7) \quad \|S_n^v\|_{L^\infty}, \|S_n^B\|_{L^\infty} \leq \|S_n^v(0)\|_{L^\infty} + \|S_n^B(0)\|_{L^\infty} + \varepsilon \int_0^t \|\mathbf{S}^\varphi \mathcal{V}^m\|^2 + (1 + \varepsilon) \int_0^t \Lambda_{\infty, m}$$

9.2. first order estimate. About first order estimate, we basically follow argument of [1]. We divide thin layer near the boundary of S, then can apply sobolev embedding to lower part, since it lose essential information about conormal stuff. Main part is L^∞ estimate for $\|\chi Z S_n^v\|_{L^\infty}$. Here, we know that direct maximal principle of transport equation is not good way since commutator between Z and Δ^φ . [1] introduces new transformation. Let's define transformation Ψ ,

$$(9.8) \quad \begin{aligned} \Psi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) &\rightarrow \Omega_t \\ x = (y, z) &\mapsto \begin{pmatrix} y \\ h(t, y) \end{pmatrix} + z \mathbf{n}^b(t, y) \end{aligned}$$

where \mathbf{n}^b is unit normal at the boundary, $(-\nabla h, 1)/|\mathbf{N}|$. To show that this is diffeomorphism near the boundary, we check

$$D\Psi(t, \cdot) = \begin{pmatrix} 1 & 0 & -\partial_1 h \\ 0 & 1 & -\partial_2 h \\ \partial_1 h & \partial_2 h & 1 \end{pmatrix} + \begin{pmatrix} -z\partial_{11}h & -z\partial_{12}h & 0 \\ -z\partial_{21}h & -z\partial_{22}h & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is diffeomorphism near the boundary since norm of second matrix is controlled by $|h|_{2, \infty}$. So, we restrict $\Psi(t, \cdot)$ on $\mathbb{R}^2 \times (-\delta, 0)$ so that it is diffeomorphism. (δ is depend on c_0 . Of course, think that above support separation was done by $\chi(z) = \kappa(\frac{z}{\delta(c_0)})$). Now we write laplacian Δ^φ with respect to Riemannian metric of above parametrization. Riemannian metric becomes,

$$(9.9) \quad g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}$$

where \tilde{g} is 2×2 block matrix. And with this metric, laplacian becomes,

$$(9.10) \quad \Delta_g f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\tilde{g}} f$$

where

$$\Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1 \leq i, j \leq 2} \partial_{y^i} (\tilde{g}^{ij} |\tilde{g}|^{\frac{1}{2}} \partial_{y^j} f)$$

where \tilde{g}^{ij} is inverse matrix element of \tilde{g} . Notice that this map is invertible near the boundary (thin layer of thickness δ which depends on c_0 .) And then as like in [1], we localize $\mathbf{S}^\varphi v$ by multiplying $\chi(z) = \kappa(\frac{z}{\delta(c_0)})$, that means this is 1 at thin layer and then smoothly decay to zero. We define this as

$$(9.11) \quad S_v^\chi = \chi(z) \mathbf{S}^\varphi v, \quad S_B^\chi = \chi(z) \mathbf{S}^\varphi B$$

We find the equation for the S_v^χ, S_B^χ as follow,

$$(9.12) \quad \partial_t^\varphi S_v^\chi + (v \cdot \nabla^\varphi) S_v^\chi - (B \cdot \nabla^\varphi) S_B^\chi - \varepsilon \Delta^\varphi (S_v^\chi) = F_{S^\chi} \doteq F^\chi + F_v$$

where

$$\begin{aligned} F^\chi &= (V_z \partial_z \chi) \mathbf{S}^\varphi v - \left(\frac{B \cdot \mathbf{N}}{\partial_z \chi} \partial_z \chi \right) \mathbf{S}^\varphi B - \varepsilon \nabla^\varphi \chi \cdot \nabla^\varphi \mathbf{S}^\varphi v - \varepsilon \Delta^\varphi \chi \mathbf{S}^\varphi v \\ F_v &= -\chi (\mathbf{D}^\varphi)^2 q - \frac{\chi}{2} ((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^T)^2) + \frac{\chi}{2} ((\nabla^\varphi B)^2 + ((\nabla^\varphi B)^T)^2) \end{aligned}$$

note that F^χ has $\mathbf{S}^\varphi v$ and $\mathbf{S}^\varphi B$ where as F_v has only non symmetric parts. Note that F^χ is supported away from the boundary because of $\nabla \chi$, and

$$\|F^\chi\|_{1, \infty} \leq \Lambda_{m, \infty}$$

For faraday law equation,

$$(9.13) \quad \partial_t^\varphi S_B^\chi + (v \cdot \nabla^\varphi) S_B^\chi - (B \cdot \nabla^\varphi) S_v^\chi - \varepsilon \Delta^\varphi (S_B^\chi) = E_{S^\chi}$$

where

$$E_{S^\chi} = (V_z \partial_z \chi) \mathbf{S}^\varphi B - \left(\frac{B \cdot \mathbf{N}}{\partial_z \varphi} \partial_z \chi \right) \mathbf{S}^\varphi v - \varepsilon \nabla^\varphi \chi \cdot \nabla^\varphi \mathbf{S}^\varphi B - \varepsilon \Delta^\varphi \chi \mathbf{S}^\varphi B$$

Note that for faraday law, equation of S_n^B is much simpler than S_n^v and we don't have F_v type commutators. Note that E_{S^χ} is supported away from the boundary because of $\nabla \chi$, and

$$\|E_{S^\chi}\|_{1,\infty} \leq \Lambda_{m,\infty}$$

Now we define \tilde{S}_v , and S_v^Ψ by

$$\begin{aligned} \tilde{S} \\ \Phi : S \rightarrow \Omega_t \leftarrow S : \Psi \end{aligned}$$

Left S is original space. Now define, \tilde{S}_v , \tilde{S}_B in Ω_t and S_v^χ , S_B^χ on left S (which is just localized $\mathbf{S}^\varphi(v)$, $\mathbf{S}^\varphi(B)$), and $S_{v,B}^\Psi$ on right S . $S_{v,B}^\Psi$ is our main stuff to estimate and this measure $S_v^\chi = \mathbf{S}_{loc}^\varphi(v)$ and $S_v^\chi = \mathbf{S}_{loc}^\varphi(B)$ in corresponding point. i.e

$$(9.14) \quad S_{v,B}^\Psi(t, y, z) = \tilde{S}_{v,B}(t, \Psi(t, y, z)) = S_{v,B}^\chi(t, (\Phi^{-1} \circ \Psi)(t, y, z))$$

So \tilde{S}_v and \tilde{S}_B solves (note that φ come from Φ so \tilde{S} solves similar equation in original domain.)

$$(9.15) \quad \partial_t \tilde{S}_v + (u \cdot \nabla) \tilde{S}_v - (H \cdot \nabla) \tilde{S}_B - \varepsilon \Delta \tilde{S}_v = F_{S^\chi}(t, \Phi^{-1}(t, \cdot))$$

$$(9.16) \quad \partial_t \tilde{S}_B + (u \cdot \nabla) \tilde{S}_B - (H \cdot \nabla) \tilde{S}_v - \varepsilon \Delta \tilde{S}_v = E_{S^\chi}(t, \Phi^{-1}(t, \cdot))$$

We use Laplacian (9.9) to transform above equation via Ψ . Then S_v^Ψ , S_B^Ψ solves,

$$(9.17) \quad \partial_t S_v^\Psi + (\omega_v \cdot \nabla) S_v^\Psi - (\omega_B \cdot \nabla) S_B^\Psi - \varepsilon \left(\partial_{zz} S_v^\Psi + \frac{1}{2} \partial_z (\ln |g|) \partial_z S_v^\Psi + \Delta_{\tilde{g}} S_v^\Psi \right) = F_{S^\chi}(t, (\Phi^{-1} \circ \Psi)(t, \cdot))$$

$$(9.18) \quad \partial_t S_B^\Psi + (\omega_v \cdot \nabla) S_B^\Psi - (\omega_B \cdot \nabla) S_v^\Psi - \varepsilon \left(\partial_{zz} S_B^\Psi + \frac{1}{2} \partial_z (\ln |g|) \partial_z S_B^\Psi + \Delta_{\tilde{g}} S_B^\Psi \right) = E_{S^\chi}(t, (\Phi^{-1} \circ \Psi)(t, \cdot))$$

where

$$\begin{aligned} \omega_v &= \bar{\chi}(\mathbf{D}\Psi)^{-1} (v(t, \Phi^{-1} \circ \Psi) - \partial_t \Psi) \\ \omega_B &= \bar{\chi}(\mathbf{D}\Psi)^{-1} B(t, \Phi^{-1} \circ \Psi) \end{aligned}$$

S^Ψ is compactly supported near the boundary and function $\bar{\chi}$ is slightly larger support in z such that $\bar{\chi} S^\Psi = S^\Psi$. Note that this function allows us to have ω which is also supported near the boundary. Now we set the alternative for S_n^v and S_n^B , which are $S_{v,n}^\Psi, S_{B,n}^\Psi$

$$(9.19) \quad S_{v,n}^\Psi = \Pi^b(t, y) S_v^\Psi \mathbf{n}^b(t, y)$$

$$S_{B,n}^\Psi = \Pi^b(t, y) S_B^\Psi \mathbf{n}^b(t, y)$$

where $\Pi^b = \mathbf{I} - \mathbf{n}^b \otimes \mathbf{n}^b$, which is tangential projection the boundary (previous projection was about $\mathbf{n} = (-\nabla \varphi, 1)$, where now is about $\mathbf{n}^b = (-\nabla h, 1)$) We get the equation for $S_{v,n}^\Psi, S_{B,n}^\Psi$

$$(9.20) \quad \partial_t S_{v,n}^\Psi + (\omega_v \cdot \nabla) S_{v,n}^\Psi - (\omega_B \cdot \nabla) S_{B,n}^\Psi - \varepsilon \left(\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \right) S_{v,n}^\Psi = F_n^\Psi$$

where

$$F_n^\Psi = (\Pi^b F_{S^\chi} \mathbf{n}^b + F_n^{\Psi,1} + F_n^{\Psi,2} + F_n^{\Psi,3})$$

with

$$\begin{aligned} F_n^{\Psi,1} &= ((\partial_t + \omega_{v,y} \cdot \nabla_y) \Pi^b) S_v^\Psi \mathbf{n}^b + \Pi^b S_v^\Psi (\partial_t + \omega_{v,y} \cdot \nabla_y) \mathbf{n}^b \\ F_n^{\Psi,2} &= -\varepsilon \Pi^b (\Delta_{\tilde{g}} S_v^\Psi) \mathbf{n}^b \\ F_n^{\Psi,3} &= ((\omega_{B,y} \cdot \nabla_y) \Pi^b) S_B^\Psi \mathbf{n}^b + \Pi^b S_B^\Psi (\omega_{B,y} \cdot \nabla_y) \mathbf{n}^b \end{aligned}$$

$$(9.21) \quad \partial_t S_{B,n}^\Psi + (\omega_v \cdot \nabla) S_{B,n}^\Psi - (\omega_B \cdot \nabla) S_{v,n}^\Psi - \varepsilon \left(\partial_{zz} S_{B,n}^\Psi + \frac{1}{2} \partial_z (\ln |g|) \partial_z \right) S_{B,n}^\Psi = E_n^\Psi$$

where

$$E_n^\Psi = (\Pi^b E_{S^\Psi} \mathbf{n}^b + E_n^{\Psi,1} + E_n^{\Psi,2} + E_n^{\Psi,3})$$

with

$$\begin{aligned} E_n^{\Psi,1} &= ((\partial_t + \omega_{v,y} \cdot \nabla_y) \Pi^b) S_B^\Psi \mathbf{n}^b + \Pi^b S_B^\Psi (\partial_t + \omega_{v,y} \cdot \nabla_y) \mathbf{n}^b \\ E_n^{\Psi,2} &= -\varepsilon \Pi^b (\triangle_{\tilde{g}} S_B^\Psi) \mathbf{n}^b \\ E_n^{\Psi,3} &= ((\omega_{B,y} \cdot \nabla_y) \Pi^b) S_v^\Psi \mathbf{n}^b + \Pi^b S_v^\Psi (\omega_{B,y} \cdot \nabla_y) \mathbf{n}^b \end{aligned}$$

and boundary condition,

$$(9.22) \quad S_{v,n}^\Psi|_{z=0} = 0, \quad S_{B,n}^\Psi|_{z=0} = 0$$

Since $S_{v,n}^\Psi = \Pi S^\varphi v \mathbf{n} \doteq S_n^v$ and $S_{B,n}^\Psi = \Pi S^\varphi B \mathbf{n} \doteq S_n^B$ of which boundary condition is already given. Now we state the lemma 9.5 in [1].

Lemma 9.2. *Consider $\mathcal{T} : S \rightarrow S$ such that $\mathcal{T}(y, 0) = y$, $\forall y \in \mathbb{R}^2$ and let $g(x) = f(\mathcal{T}x)$. Then for every $k \geq 1$, we have the estimate*

$$\|g\|_{k,\infty} \leq \Lambda (\|\nabla \mathcal{T}\|_{k-1,\infty}) \|f\|_{k,\infty}$$

Remark 9.3. Meaning of this lemma : Sobolev conormal spaces are invariant by diffeomorphism which preserve the boundary. i.e If f is conormal k, ∞ , then $f \circ \mathcal{T}$ is also conormal k, ∞ if \mathcal{T} preserves the boundary. Similar holds for $\|\cdot\|_m$ type sobolev space.

We use this lemma (and remark) to show that equivalency of S_n^v and $S_{v,n}^\Psi$. (also for B) From [1] we have equivalency,

$$(9.23) \quad \|S_{v,n}^\Psi\|_{1,\infty} \leq \Lambda_0 (\|S_n^v\|_{1,\infty} + \|v\|_{2,\infty}), \quad \|S_n^v\|_{1,\infty} \leq \Lambda_0 \left(\|S_{v,n}^\Psi\|_{1,\infty} + \Lambda \left(\frac{1}{c_0}, |h|_m + \|\mathcal{V}^m\| + \|S_n^v\|_{m-2} \right) \right)$$

$$\|S_{B,n}^\Psi\|_{1,\infty} \leq \Lambda_0 (\|S_n^B\|_{1,\infty} + \|B\|_{2,\infty}), \quad \|S_n^B\|_{1,\infty} \leq \Lambda_0 \left(\|S_{B,n}^\Psi\|_{1,\infty} + \Lambda \left(\frac{1}{c_0}, |h|_m + \|\mathcal{B}^m\| + \|S_n^B\|_{m-2} \right) \right)$$

Note that proposition 9.1 in [1] is also true for B , since it come from sobolev embedding and alinhac's structure. Meanwhile we should get estimate for $\omega_{v,B}$. Using same argument,

$$(9.24) \quad \|\omega_v\|_{1,\infty} \leq \Lambda \left(\frac{1}{c_0}, |h|_{3,\infty} + \|v\|_{1,\infty} + \|\partial_t \Psi\|_{1,\infty} \right) \leq \Lambda \left(\frac{1}{c_0}, |h|_{3,\infty} + \|v\|_{2,\infty} + |\partial_t h|_{2,\infty} \right) \leq \Lambda_{\infty,m}$$

$$\|\omega_B\|_{1,\infty} \leq \Lambda \left(\frac{1}{c_0}, |h|_{3,\infty} + \|B\|_{1,\infty} \right) \leq \Lambda \left(\frac{1}{c_0}, |h|_{3,\infty} + \|v\|_{2,\infty} \right) \leq \Lambda_{\infty,m}$$

Now we make proposition about estimate of $\|Z_i S_{v,n}^\Psi\|_{L^\infty}$, $\|Z_i S_{B,n}^\Psi\|_{L^\infty}$. Note that $\|S_{v,n}^\Psi\|_{L^\infty}$, $\|S_{B,n}^\Psi\|_{L^\infty}$ is already given above by just maximal principle and adding,deduction method.

Proposition 9.4. *We have the estimate.*

$$\begin{aligned} (9.25) \quad & \|Z_3 S_n^v(t)\|_{L^\infty}, \|Z_3 S_n^B(t)\|_{L^\infty} \leq \Lambda_0 (\|v(0)\|_{E^{2,\infty}} + \|B(0)\|_{E^{2,\infty}}) \\ & + \Lambda \left(\frac{1}{c_0}, \|\mathcal{V}^m\| + \|\mathcal{B}^m\| + \|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_m \right) \\ & + \int_0^t \Lambda_{\infty,m} (1 + \|S_n^v\|_{1,\infty} + \|S_n^B\|_{1,\infty} + \varepsilon \|\nabla \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|B\|_{4,\infty}) \end{aligned}$$

Proof. 1) When $i = 1, 2$, we apply ∂_i to get these two.

$$\begin{aligned} (9.26) \quad & \partial_t \partial_i S_{v,n}^\Psi + (\omega_v \cdot \nabla) \partial_i S_{v,n}^\Psi - (\omega_B \cdot \nabla) \partial_i S_{B,n}^\Psi - \varepsilon \left(\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \right) \partial_i S_{v,n}^\Psi \\ & = \partial_i F_n^\Psi - \partial_i \omega_v \cdot \nabla S_{v,n}^\Psi + \partial_i \omega_B \cdot \nabla S_{B,n}^\Psi - \frac{\varepsilon}{2} \partial_z S_{v,n}^\Psi \partial_{iz}^2 (\ln |g|) \end{aligned}$$

$$\begin{aligned} (9.27) \quad & \partial_t \partial_i S_{B,n}^\Psi + (\omega_v \cdot \nabla) \partial_i S_{B,n}^\Psi - (\omega_B \cdot \nabla) \partial_i S_{v,n}^\Psi - \varepsilon \left(\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \right) \partial_i S_{B,n}^\Psi \\ & = \partial_i E_n^\Psi - \partial_i \omega_v \cdot \nabla S_{B,n}^\Psi + \partial_i \omega_B \cdot \nabla S_{v,n}^\Psi - \frac{\varepsilon}{2} \partial_z S_{B,n}^\Psi \partial_{iz}^2 (\ln |g|) \end{aligned}$$

We also add and deduct these two equations.

$$(9.28) \quad \begin{aligned} & \partial_t \partial_i (S_{v,n}^\Psi + S_{B,n}^\Psi) + ((\omega_v - \omega_B) \cdot \nabla) \partial_i (S_{v,n}^\Psi + S_{B,n}^\Psi) - \varepsilon \left(\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \right) \partial_i (S_{v,n}^\Psi + S_{B,n}^\Psi) \\ &= \partial_i (F_n^\Psi + E_n^\Psi) + (\partial_i \omega_B - \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi + S_{B,n}^\Psi) - \frac{\varepsilon}{2} \partial_z (S_{v,n}^\Psi + S_{B,n}^\Psi) \partial_{iz}^2 (\ln |g|) \end{aligned}$$

$$(9.29) \quad \begin{aligned} & \partial_t \partial_i (S_{v,n}^\Psi - S_{B,n}^\Psi) + ((\omega_v + \omega_B) \cdot \nabla) \partial_i (S_{v,n}^\Psi - S_{B,n}^\Psi) - \varepsilon \left(\partial_{zz} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \right) \partial_i (S_{v,n}^\Psi - S_{B,n}^\Psi) \\ &= \partial_i (F_n^\Psi - E_n^\Psi) - (\partial_i \omega_B + \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi - S_{B,n}^\Psi) - \frac{\varepsilon}{2} \partial_z (S_{v,n}^\Psi - S_{B,n}^\Psi) \partial_{iz}^2 (\ln |g|) \end{aligned}$$

Maximal principle becomes,

$$(9.30) \quad \begin{aligned} & \|\partial_i (S_{v,n}^\Psi + S_{B,n}^\Psi)(t)\|_{L^\infty} \leq \|\partial_i (S_{v,n}^\Psi + S_{B,n}^\Psi)(0)\|_{L^\infty} \\ &+ \int_0^t (\|\partial_i (F_n^\Psi + E_n^\Psi)\|_{L^\infty} + \|(\partial_i \omega_B - \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi + S_{B,n}^\Psi)\|_{L^\infty} + \varepsilon \|\partial_z (S_{v,n}^\Psi + S_{B,n}^\Psi) \partial_{iz}^2 (\ln |g|)\|_{L^\infty}) \\ &\leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t \left(\|\partial_i F_n^\Psi\|_{L^\infty} + \|\partial_i E_n^\Psi\|_{L^\infty} + \|(\partial_i \omega_B - \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi + S_{B,n}^\Psi)\|_{L^\infty} + \varepsilon \Lambda\left(\frac{1}{c_0}, |h|_{3,\infty}\right) (\|\partial_z S_{v,n}^\Psi\|_{L^\infty} + \|\partial_z S_{B,n}^\Psi\|_{L^\infty}) \right) \\ &\leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t (\|\partial_i F_n^\Psi\|_{L^\infty} + \|\partial_i E_n^\Psi\|_{L^\infty} + \|(\partial_i \omega_B - \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi + S_{B,n}^\Psi)\|_{L^\infty} + \Lambda_{\infty,m}) \\ (9.31) \quad & \|\partial_i (S_{v,n}^\Psi - S_{B,n}^\Psi)(t)\|_{L^\infty} \leq \|\partial_i (S_{v,n}^\Psi - S_{B,n}^\Psi)(0)\|_{L^\infty} \\ &+ \int_0^t (\|\partial_i (F_n^\Psi - E_n^\Psi)\|_{L^\infty} + \|(\partial_i \omega_B + \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi - S_{B,n}^\Psi)\|_{L^\infty} + \varepsilon \|\partial_z (S_{v,n}^\Psi - S_{B,n}^\Psi) \partial_{iz}^2 (\ln |g|)\|_{L^\infty}) \\ &\leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t \left(\|\partial_i F_n^\Psi\|_{L^\infty} + \|\partial_i E_n^\Psi\|_{L^\infty} + \|(\partial_i \omega_B + \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi - S_{B,n}^\Psi)\|_{L^\infty} + \varepsilon \Lambda\left(\frac{1}{c_0}, |h|_{3,\infty}\right) (\|\partial_z S_{v,n}^\Psi\|_{L^\infty} + \|\partial_z S_{B,n}^\Psi\|_{L^\infty}) \right) \\ &\leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t (\|\partial_i F_n^\Psi\|_{L^\infty} + \|\partial_i E_n^\Psi\|_{L^\infty} + \|(\partial_i \omega_B + \partial_i \omega_v) \cdot \nabla (S_{v,n}^\Psi - S_{B,n}^\Psi)\|_{L^\infty} + \Lambda_{\infty,m}) \end{aligned}$$

I) $\|\partial_i \omega_v \cdot \nabla S_{v,n}^\Psi\|_{L^\infty}$ estimate.

$$\|\partial_i \omega_v \cdot \nabla S_{v,n}^\Psi\|_{L^\infty} \leq \|\omega_v\|_{1,\infty} \|S_{v,n}^\Psi\|_{1,\infty} + \|\partial_i \omega_{v,3} \partial_z S_{v,n}^\Psi\|_{L^\infty} \leq \Lambda_{\infty,m} + \|\partial_i \omega_{v,3} \partial_z S_{v,n}^\Psi\|_{L^\infty}$$

Now note that,

$$w^b = (\mathbf{D}\Phi(t, y, 0))^{-1} \begin{pmatrix} v^b \\ -\partial_t h \end{pmatrix}$$

and

$$\omega_{v,3}^b = \frac{1}{|\mathbf{N}|} (v^b \cdot \mathbf{N} - \partial_t h) = 0$$

by boundary condition. So $\partial_i \omega_{v,3}$ also vanishes on the boundary since $i = 1, 2$. Then we get the estimate of the last term of above,

$$(9.32) \quad \begin{aligned} & \|\partial_i \omega_{v,3} \partial_z S_{v,n}^\Psi\|_{L^\infty} \leq \left\| \frac{1-z}{z} \partial_i \omega_{v,3} \frac{z}{1-z} \partial_z S_{v,n}^\Psi \right\|_{L^\infty} \leq \left\| \frac{1-z}{z} \partial_i \omega_{v,3} \right\|_{L^\infty} \|S_{v,n}^\Psi\|_{1,\infty} \\ &\leq \left\| \frac{1-z}{z} (0 + |\partial_z \partial_i \omega_{v,3}|_{L_{z,loc}^\infty} z) \right\|_{L^\infty} \|S_{v,n}^\Psi\|_{1,\infty} \leq \|\partial_z \partial_i \omega_{v,3}\|_{L^\infty} \|S_{v,n}^\Psi\|_{1,\infty} \leq \Lambda \left(\frac{1}{c_0}, |h|_{3,\infty} + \|v\|_{E^{2,\infty}} \right) \leq \Lambda_{\infty,m} \end{aligned}$$

Hence

$$\|\partial_i \omega_v \cdot \nabla S_{v,n}^\Psi\|_{L^\infty} \leq \Lambda_{\infty,m}$$

Similarly, we get

$$\|\partial_i \omega_v \cdot \nabla S_{B,n}^\Psi\|_{L^\infty} \leq \Lambda_{\infty,m}$$

II) $\|\partial_i \omega_B \cdot \nabla S_{B,n}^\Psi\|_{L^\infty}$ estimate.

$$\|\partial_i \omega_B \cdot \nabla S_{B,n}^\Psi\|_{L^\infty} \leq \|\omega_B\|_{1,\infty} \|S_{B,n}^\Psi\|_{1,\infty} + \|\partial_i \omega_{B,3} \partial_z S_{B,n}^\Psi\|_{L^\infty} \leq \Lambda_{\infty,m} + \|\partial_i \omega_{B,3} \partial_z S_{B,n}^\Psi\|_{L^\infty}$$

As like above I), $\omega_B^b = 0$ since B vanish on the boundary, so

$$\omega_B^b = (\mathbf{D}\Psi(t, y, 0))^{-1}(B^b) = 0$$

Now very similarly, using zero boundary value property of $\partial_i \omega_B^b$, we get

$$\|\partial_i \omega_B \cdot \nabla S_{B,n}^\Psi\|_{L^\infty} \leq \Lambda_{\infty,m}$$

Similarly, we get

$$\|\partial_i \omega_B \cdot \nabla S_{v,n}^\Psi\|_{L^\infty} \leq \Lambda_{\infty,m}$$

III) $\|\partial_i F_n^\Psi\|_{L^\infty}$ estimate.

$$\partial_i F_n^\Psi = \partial_i (\Pi^b F_{S^\times} \mathbf{n}^b) + \partial_i F_n^{\Psi,1} + \partial_i F_n^{\Psi,2} + \partial_i F_n^{\Psi,3}$$

$$\|F_n^{\Psi,1}\|_{1,\infty}, \|F_n^{\Psi,3}\|_{1,\infty} \leq \Lambda_{\infty,m}$$

$$\|F_n^{\Psi,2}\|_{1,\infty} \leq \varepsilon \Lambda_{\infty,m} (\|S_n^v\|_{3,\infty} + \|v\|_{4,\infty})$$

Considering $\partial_i (\Pi^b F_{S^\times} \mathbf{n}^b)$ (among this, the only thing to care is pressure part), we get

$$\|F_n^\Psi\|_{1,\infty} \leq \Lambda_{\infty,m} (1 + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \|\Pi^b ((\mathbf{D}^\varphi)^2 q) \mathbf{n}^b\|_{1,\infty})$$

$$\begin{aligned} \|\Pi^b ((\mathbf{D}^\varphi)^2 q) \mathbf{n}^b\|_{1,\infty} &\leq \Lambda_0 (\|\nabla q^E\|_{2,\infty} + \|\nabla q^{NS}\|_{2,\infty}) \\ &\leq \Lambda_0 (1 + \varepsilon \|\mathbf{S}^\varphi \mathcal{V}^m\|) \end{aligned}$$

Hence,

$$\|F_n^\Psi\|_{1,\infty} \leq \Lambda_{\infty,m} (1 + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \varepsilon \|\nabla \mathcal{V}^m\|)$$

IV) $\|\partial_i E_n^\Psi\|_{L^\infty}$ estimate.

$$\partial_i E_n^\Psi = \partial_i (\Pi^b E_{S^\times} \mathbf{n}^b) + \partial_i E_n^{\Psi,1} + \partial_i E_n^{\Psi,2} + \partial_i E_n^{\Psi,3}$$

$$\|E_n^{\Psi,1}\|_{1,\infty}, \|E_n^{\Psi,3}\|_{1,\infty} \leq \Lambda_{\infty,m}$$

$$\|E_n^{\Psi,2}\|_{1,\infty} \leq \lambda \Lambda_{\infty,m} (\|S_n^B\|_{3,\infty} + \|B\|_{4,\infty})$$

Considering $\partial_i (\Pi^b E_{S^\times} \mathbf{n}^b)$ (These is no terms like F_v , so no terms of pressure), we get simply

$$\|E_n^\Psi\|_{1,\infty} \leq \Lambda_{\infty,m} (1 + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|B\|_{4,\infty})$$

Considering all estimates, we get

$$\begin{aligned} (9.33) \quad &\|\partial_i (S_{v,n}^\Psi + S_{B,n}^\Psi)(t)\|_{L^\infty} \leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t \Lambda_{\infty,m} (1 + \varepsilon \|\nabla \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \varepsilon \|B\|_{4,\infty}) \end{aligned}$$

$$\begin{aligned} (9.34) \quad &\|\partial_i (S_{v,n}^\Psi - S_{B,n}^\Psi)(t)\|_{L^\infty} \leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t \Lambda_{\infty,m} (1 + \varepsilon \|\nabla \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \varepsilon \|B\|_{4,\infty}) \end{aligned}$$

Hence as we did in zero-order estimate, we get

$$\begin{aligned} (9.35) \quad &\|\partial_i S_{v,n}^\Psi(t)\|_{L^\infty}, \|\partial_i S_{B,n}^\Psi(t)\|_{L^\infty} \leq \|\partial_i S_{v,n}^\Psi(0)\|_{L^\infty} + \|\partial_i S_{B,n}^\Psi(0)\|_{L^\infty} \\ &+ \int_0^t \Lambda_{\infty,m} (1 + \varepsilon \|\nabla \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \varepsilon \|B\|_{4,\infty}) \end{aligned}$$

2) When $i = 3$, we apply $Z_3 = \frac{z}{1-z} \partial_z$. commutator between $Z_3, \varepsilon \partial_{zz}$ should be treated carefully. As like in [1], its convenient to eliminate $\varepsilon \partial_z (\ln |g|) \partial_z$ in modified laplacian. This is done by setting,

$$\begin{aligned} (9.36) \quad &\rho_v(t, y, z) = |g|^{\frac{1}{4}} S_{v,n}^\Psi = |g|^{\frac{1}{4}} \Pi^b S_v^\Psi \mathbf{n}^b \\ &\rho_B(t, y, z) = |g|^{\frac{1}{4}} S_{B,n}^\Psi = |g|^{\frac{1}{4}} \Pi^b S_B^\Psi \mathbf{n}^b \end{aligned}$$

since this solves,

$$(9.37) \quad \partial_t \rho_v + \omega_v \cdot \nabla \rho_v - \omega_B \cdot \nabla \rho_B - \varepsilon \partial_{zz} \rho_v = |g|^{\frac{1}{4}} (F_n^\Psi + F_g) \doteq \mathcal{H}_1$$

where

$$F_g = \frac{\rho_v}{|g|^{\frac{1}{2}}} (\omega_v \cdot \nabla - \varepsilon \partial_{zz}) |g|^{\frac{1}{4}} + \frac{\rho_B}{|g|^{\frac{1}{2}}} (\omega_B \cdot \nabla) |g|^{\frac{1}{4}}$$

and for faraday,

$$(9.38) \quad \partial_t \rho_B + \omega_v \cdot \nabla \rho_B - \omega_B \cdot \nabla \rho_v - \varepsilon \partial_{zz} \rho_B = |g|^{\frac{1}{4}} (E_n^\Psi + E_g) \doteq \mathcal{H}_2$$

where

$$E_g = \frac{\rho_B}{|g|^{\frac{1}{2}}} (\omega_v \cdot \nabla - \varepsilon \partial_{zz}) |g|^{\frac{1}{4}} + \frac{\rho_v}{|g|^{\frac{1}{2}}} (\omega_B \cdot \nabla) |g|^{\frac{1}{4}}$$

We treat ρ_v instead of $Z_3 S_{v,n}^\Psi$ (same for B) since we have equivalency

$$(9.39) \quad \|Z_3 S_{v,n}^\Psi\|_{L^\infty} \leq \Lambda_0 \|\rho_v\|_{1,\infty}, \quad \|\rho_v\|_{1,\infty} \leq \Lambda_0 \|S_{v,n}^\Psi\|_{1,\infty}$$

$$\|Z_3 S_{B,n}^\Psi\|_{L^\infty} \leq \Lambda_0 \|\rho_B\|_{1,\infty}, \quad \|\rho_B\|_{1,\infty} \leq \Lambda_0 \|S_{B,n}^\Psi\|_{1,\infty}$$

So equivalency of $\|S_{v,n}^\Psi\|_{1,\infty}$ and $\|\rho_v\|_{1,\infty}$ (same for B .) Note that ρ_v, ρ_B are also zero on the boundary $z = 0$. Now we state Lemma 9.6 in [1]. Note that for convection ω , the third component ω_3 vanishes on the boundary.

Lemma 9.5. *For smooth function ρ ,*

$$(9.40) \quad \partial_t \rho + \omega \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{H}$$

where ω_3 vanishes on the boundary. Assume ρ and \mathcal{H} are compactly supported in z , then we have,

$$\|Z_i \rho(t)\|_{L^\infty} \leq \|Z_i \rho_0\|_{L^\infty} + \|\rho_0\|_{L^\infty} + \int_0^t ((\|\omega\|_{E^{2,\infty}} + \|\partial_{zz} \omega_3\|_{L^\infty})(\|\rho\|_{1,\infty} + \|\rho\|_4) + \|\mathcal{H}\|_{1,\infty}), \quad i = 1, 2, 3$$

By adding and deducting ρ equations we have

$$(9.41) \quad \partial_t (\rho_v + \rho_B) + (\omega_v - \omega_B) \cdot \nabla (\rho_v + \rho_B) - \varepsilon \partial_{zz} (\rho_v + \rho_B) = \mathcal{H}_1 + \mathcal{H}_2$$

$$(9.42) \quad \partial_t (\rho_v - \rho_B) + (\omega_v + \omega_B) \cdot \nabla (\rho_v - \rho_B) - \varepsilon \partial_{zz} (\rho_v - \rho_B) = \mathcal{H}_1 - \mathcal{H}_2$$

Using this lemma,

$$(9.43) \quad \|Z_3 (\rho_v + \rho_B)(t)\|_{L^\infty} \leq \|Z_3 (\rho_v + \rho_B)(0)\|_{L^\infty} + \|(\rho_v + \rho_B)(0)\|_{L^\infty} \\ + \int_0^t ((\|\omega_v - \omega_B\|_{E^{2,\infty}} + \|\partial_{zz} (\omega_v - \omega_B)_3\|_{L^\infty})(\|\rho_v + \rho_B\|_{1,\infty} + \|\rho_v + \rho_B\|_4) + \|\mathcal{H}_1\|_{1,\infty} + \|\mathcal{H}_2\|_{1,\infty})$$

$$(9.44) \quad \|Z_3 (\rho_v - \rho_B)(t)\|_{L^\infty} \leq \|Z_3 (\rho_v - \rho_B)(0)\|_{L^\infty} + \|(\rho_v - \rho_B)(0)\|_{L^\infty} \\ + \int_0^t ((\|\omega_v + \omega_B\|_{E^{2,\infty}} + \|\partial_{zz} (\omega_v + \omega_B)_3\|_{L^\infty})(\|\rho_v - \rho_B\|_{1,\infty} + \|\rho_v - \rho_B\|_4) + \|\mathcal{H}_1\|_{1,\infty} + \|\mathcal{H}_2\|_{1,\infty})$$

I) $\|\mathcal{H}_1\|_{1,\infty}, \|\mathcal{H}_1\|_{2,\infty}$ estimate.

$$\|\mathcal{H}_1\|_{1,\infty} \leq \|F_n^\Psi\|_{1,\infty} + \|F_g\|_{1,\infty}, \quad \|\mathcal{H}_2\|_{1,\infty} \leq \|E_n^\Psi\|_{1,\infty} + \|E_g\|_{1,\infty}$$

We already estimated $\|F_n^\Psi\|_{1,\infty}, \|E_n^\Psi\|_{1,\infty}$ above. So,

$$\|\mathcal{H}_1\|_{1,\infty} \leq \Lambda_{\infty,m} (1 + \varepsilon \|\mathbf{S}^\varphi \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|v\|_{4,\infty})$$

$$\|\mathcal{H}_2\|_{1,\infty} \leq \Lambda_{\infty,m} (1 + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|B\|_{4,\infty})$$

II) $\|\rho_v\|_4, \|\rho_B\|_4$ estimate.

$$\|\rho_v\|_4 \leq \Lambda \left(\frac{1}{c_0}, |h|_6 + \|S_{v,n}^\Psi\|_4 \right) \leq \Lambda_{\infty,m}$$

$$\|\rho_B\|_4 \leq \Lambda \left(\frac{1}{c_0}, |h|_6 + \|S_{B,n}^\Psi\|_4 \right) \leq \Lambda_{\infty,m}$$

III) $\|\omega_v\|_{E^{2,\infty}}, \|\omega_B\|_{E^{2,\infty}}$ estimate. From definition,

$$\|\omega_v\|_{E^{2,\infty}}, \|\omega_B\|_{E^{2,\infty}} \leq \Lambda_{\infty,m}$$

IV) $\|\partial_{zz}\omega_{v,3}\|_{L^\infty}, \|\partial_{zz}\omega_{B,3}\|_{L^\infty}$ estimate. It looks that this term has two normal derivatives, so should be careful. First,

$$\|\partial_{zz}(\bar{\chi}(\mathbf{D}\Psi^{-1}\partial_t\Psi))\|_{L^\infty} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |\partial_t h|_{2,\infty}\right) \leq \Lambda_{\infty,m}$$

Main part is

$$\|\partial_{zz}(\bar{\chi}D\Psi^{-1}v(t, \Phi^{-1} \circ \Psi))_3\|_{L^\infty}$$

Key point is that this is bounded by term with $D\Psi^{-1}$ is estimated at $z = 0$, like

$$\leq \|\bar{\chi}\partial_{zz}((D\Phi(t, y, 0))^{-1}v(t, \Phi^{-1} \circ \Psi))_3\|_{L^\infty} + \Lambda_{\infty,m}$$

then by (9.35),

$$\leq \|\bar{\chi}\partial_{zz}(v(t, \Phi^{-1} \circ \Psi) \cdot \mathbf{n}^b)\|_{L^\infty} + \Lambda_{\infty,m}$$

we write $v(t, \Phi^{-1} \circ \Psi) = u(t, \Psi) \doteq u^\Psi(t, y, z)$ then using divergence free condition of u , we can change 1-normal derivative to tangential derivative so that $\partial_{zz} \rightarrow \partial_{iz}$ to be controlled by $\Lambda_{\infty,m}$. See more detail in [1]. At result, from [1]

$$\|\partial_{zz}\omega_{v,3}\|_{L^\infty}, \|\partial_{zz}\omega_{B,3}\|_{L^\infty} \leq \Lambda_{\infty,m}$$

So we get the estimate for $\|Z_3(\rho_v + \rho_B)\|_{L^\infty}, \|Z_3(\rho_v - \rho_B)\|_{L^\infty}$ in the same form.

$$(9.45) \quad \|Z_3(\rho_v + \rho_B)\|_{L^\infty}, \|Z_3(\rho_v - \rho_B)\|_{L^\infty} \leq \|Z_3(\rho_v + \rho_B)(0)\|_{L^\infty} \\ + \int_0^t \Lambda_{\infty,m} (1 + \|\rho_v\|_{1,\infty} + \|\rho_B\|_{1,\infty} + \varepsilon\|\mathbf{S}^\varphi\mathcal{V}^m\| + \varepsilon\|S_n^v\|_{3,\infty} + \varepsilon\|v\|_{4,\infty} + \varepsilon\|S_n^B\|_{3,\infty} + \varepsilon\|B\|_{4,\infty})$$

with

$$\|Z_3(\rho_v + \rho_B)(0)\|_{L^\infty} \leq \Lambda_0(\|v(0)\|_{E^{2,\infty}} + \|B(0)\|_{E^{2,\infty}})$$

Now using technique again, we get the same estimate for each $\|Z_3\rho_v\|_{L^\infty}, \|Z_3\rho_B\|_{L^\infty}$.

$$(9.46) \quad \|Z_3\rho_v(t)\|_{L^\infty}, \|Z_3\rho_B(t)\|_{L^\infty} \leq \Lambda_0(\|v(0)\|_{E^{2,\infty}} + \|B(0)\|_{E^{2,\infty}}) \\ + \int_0^t \Lambda_{\infty,m} (1 + \|\rho_v\|_{1,\infty} + \|\rho_B\|_{1,\infty} + \varepsilon\|\mathbf{S}^\varphi\mathcal{V}^m\| + \varepsilon\|S_n^v\|_{3,\infty} + \varepsilon\|v\|_{4,\infty} + \varepsilon\|S_n^B\|_{3,\infty} + \varepsilon\|B\|_{4,\infty})$$

3) We now know that $\|Z_i S_{(v,B),n}^\Psi\|_{L^\infty}$ estimate, and $\|Z_3\rho_{v,B}\|_{L^\infty}$. Moreover,

$$\rho_{v,B} \sim S_{(v,B),n}^\Psi \sim S_n^{v,B}$$

with help of

$$\Lambda\left(\frac{1}{c_0}, \|\mathcal{V}, \mathcal{B}\|^m + \|S_n^{v,B}\|_{m-2} + |h|_m\right)$$

Hence put all together to get,

$$(9.47) \quad \|Z_3 S_n^v(t)\|_{L^\infty}, \|Z_3 S_n^B(t)\|_{L^\infty} \leq \Lambda_0(\|v(0)\|_{E^{2,\infty}} + \|B(0)\|_{E^{2,\infty}}) \\ + \Lambda\left(\frac{1}{c_0}, \|\mathcal{V}^m\| + \|\mathcal{B}^m\| + \|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_m\right) \\ + \int_0^t \Lambda_{\infty,m} (1 + \|S_n^v\|_{1,\infty} + \|S_n^B\|_{1,\infty} + \varepsilon\|\nabla\mathcal{V}^m\| + \varepsilon\|S_n^v\|_{3,\infty} + \varepsilon\|v\|_{4,\infty} + \varepsilon\|S_n^B\|_{3,\infty} + \varepsilon\|B\|_{4,\infty})$$

□

As far, we got zero-order estimate and first-order estimate, so whole estimate for $\|S_n^{v,B}\|_{1,\infty}$.

Proposition 9.6. *Estimate of $\|S_n^{v,B}\|_{1,\infty}$*

$$(9.48) \quad \|S_n^v(t)\|_{1,\infty}^2, \|S_n^B(t)\|_{1,\infty}^2 \leq \Lambda_0(\|S_n^v(0)\|_{1,\infty}^2 + \|S_n^B(0)\|_{1,\infty}^2) \\ + \Lambda\left(\frac{1}{c_0}, \|\mathcal{V}^m\| + \|\mathcal{B}^m\| + \|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_m\right) \\ + (1 + \varepsilon) \int_0^t \Lambda_{\infty,m} + \int_0^t (\|S_n^v\|_{1,\infty}^2 + \|S_n^B\|_{1,\infty}^2 + \varepsilon\|\nabla\mathcal{V}^m\|^2 + \varepsilon\|\nabla\mathcal{B}^m\|^2 + \varepsilon\|\nabla S_n^v\|_{m-2}^2 + \varepsilon\|\nabla S_n^B\|_{m-2}^2)$$

Proof. By considering zero-order and first order,

$$(9.49) \quad \begin{aligned} & \|S_n^v(t)\|_{1,\infty}, \|S_n^B(t)\|_{1,\infty} \leq \Lambda_0 (\|v(0)\|_{E^{2,\infty}} + \|B(0)\|_{E^{2,\infty}}) \\ & + \Lambda \left(\frac{1}{c_0}, \|\mathcal{V}^m\| + \|\mathcal{B}^m\| + \|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_m \right) \\ & + (1 + \varepsilon) \int_0^t \Lambda_{\infty,m} (1 + \|S_n^v\|_{1,\infty} + \|S_n^B\|_{1,\infty} + \varepsilon \|\nabla \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{3,\infty} + \varepsilon \|v\|_{4,\infty} + \varepsilon \|S_n^B\|_{3,\infty} + \varepsilon \|B\|_{4,\infty}) \end{aligned}$$

We treat $\varepsilon \|S_n^v\|_{3,\infty}, \varepsilon \|v\|_{4,\infty}, \varepsilon \|S_n^B\|_{3,\infty}, \varepsilon \|B\|_{4,\infty}$ more. For $\varepsilon \|S_n^v\|_{3,\infty}, \varepsilon \|S_n^B\|_{3,\infty}$, by embedding,

$$\begin{aligned} \sqrt{\varepsilon} \|S_n^v\|_{3,\infty} & \leq \Lambda_{\infty,m} + \sqrt{\varepsilon} \|\nabla \mathcal{V}^m\| + \sqrt{\varepsilon} \|\nabla S_n^v\|_{m-2} \\ \sqrt{\varepsilon} \|S_n^B\|_{3,\infty} & \leq \Lambda_{\infty,m} + \sqrt{\varepsilon} \|\nabla \mathcal{B}^m\| + \sqrt{\varepsilon} \|\nabla S_n^B\|_{m-2} \end{aligned}$$

For $\varepsilon \|v\|_{4,\infty}, \varepsilon \|B\|_{4,\infty}$,

$$\begin{aligned} \sqrt{\varepsilon} \|v\|_{4,\infty} & \leq \Lambda_{\infty,m} + \sqrt{\varepsilon} \|\nabla \mathcal{V}^m\| \\ \sqrt{\varepsilon} \|B\|_{4,\infty} & \leq \Lambda_{\infty,m} + \sqrt{\varepsilon} \|\nabla \mathcal{B}^m\| \end{aligned}$$

By replacing using above 4 estimates, and using young's inequality, we get the result.

$$(9.50) \quad \begin{aligned} & \|S_n^v(t)\|_{1,\infty}, \|S_n^B(t)\|_{1,\infty} \leq \Lambda_0 (\|v(0)\|_{E^{2,\infty}} + \|B(0)\|_{E^{2,\infty}}) + \Lambda \left(\frac{1}{c_0}, \|\mathcal{V}^m\| + \|\mathcal{B}^m\| + \|S_n^v\|_{m-2} + \|S_n^B\|_{m-2} + |h|_m \right) \\ & + (1 + \varepsilon) \int_0^t \Lambda_{\infty,m} + \int_0^t (\|S_n^v\|_{1,\infty}^2 + \|S_n^B\|_{1,\infty}^2 + \varepsilon \|\nabla \mathcal{V}^m\|^2 + \varepsilon \|\nabla \mathcal{B}^m\|^2 + \varepsilon \|\nabla S_n^v\|_{m-2}^2 + \varepsilon \|\nabla S_n^B\|_{m-2}^2) \end{aligned}$$

□

9.3. $\sqrt{\varepsilon} \|\partial_z S_n^v\|_{L^\infty}, \sqrt{\varepsilon} \|\partial_z S_n^B\|_{L^\infty}$ estimate.

Proposition 9.7. *We have the estimate for $\sqrt{\varepsilon} \|\partial_z S_n^v\|_{L^\infty}, \sqrt{\varepsilon} \|\partial_z S_n^B\|_{L^\infty}$*

$$(9.51) \quad \begin{aligned} & \varepsilon \|\partial_z \rho_v\|_{L^\infty}^2, \varepsilon \|\partial_z \rho_B\|_{L^\infty}^2 \leq \Lambda_{\infty,m}(0) \\ & + 2 \int_0^t (\varepsilon \|\nabla \mathcal{V}^m\|^2 + \varepsilon \|\nabla \mathcal{B}^m\|^2 + \varepsilon \|\nabla S_n^v\|_{m-2}^2 + \varepsilon \|\nabla S_n^B\|_{m-2}^2) + (1 + 16\sqrt{t}) \int_0^t \frac{\Lambda_{\infty,m}}{\sqrt{t-\tau}} d\tau \end{aligned}$$

Proof. This estimate corresponds to $\sqrt{\varepsilon} \|\partial_{zz} v\|_{L^\infty}$. Our strategy is to derive the estimate for $\sqrt{\varepsilon} \|\partial_z \rho_{v,B}\|_{L^\infty}$, since for both v, B ,

$$\partial_z S_n^\Psi = \Pi^b \frac{\partial}{\partial z} S^\chi(t, \Phi^{-1} \circ \Psi) \mathbf{n}^b$$

we can apply Lemma 9.5 [1], so we get similar control.

$$\|\partial_z S_n^\Psi\|_{1,\infty} \leq \Lambda_0 \|\partial_z \Pi^b S^\chi(t, \Phi^{-1} \circ \Psi) \mathbf{n}^b\|_{1,\infty}$$

Then using $|\Pi - \Pi^b| + |\mathbf{n} - \mathbf{n}^b| = \mathcal{O}(z)$,

$$\|\partial_z S_n^\Psi\|_{1,\infty} \leq \Lambda_0 (\|\partial_z S_n\|_{1,\infty} + \|v\|_{2,\infty})$$

What we need is inverse argument. since the map \mathcal{T} in Lemma 9.2 conserves boundary,

$$\|\partial_z S_n\|_{1,\infty} \leq \Lambda_0 (\|\partial_z S_n^\Psi\|_{1,\infty} + \|v\|_{2,\infty})$$

and

$$\|\partial_z \rho\|_{1,\infty} = \|\partial_z (|g|^{\frac{1}{2}} \Pi^b S^\Psi \mathbf{n}^b)\|_{1,\infty}$$

On the right hand side, ∂_z hit $|g|$ and S^Ψ and we have,

$$\|\partial_z S_n^\Psi\|_{1,\infty} \leq \Lambda_0 (\|\partial_z \rho\|_{1,\infty})$$

Hence, we get the control what we expected for both v, B .

$$(9.52) \quad \begin{aligned} \sqrt{\varepsilon} \|\partial_z S_n\|_{1,\infty} & \leq \Lambda_0 (\sqrt{\varepsilon} \|\partial_z \rho_v\|_{1,\infty} + \|v\|_{2,\infty}) \\ \sqrt{\varepsilon} \|\partial_z S_n\|_{1,\infty} & \leq \Lambda_0 (\sqrt{\varepsilon} \|\partial_z \rho_B\|_{1,\infty} + \|B\|_{2,\infty}) \end{aligned}$$

As we know, ρ_v, ρ_B solves

$$\partial_t(\rho_v + \rho_B) + (\omega_v - \omega_B) \cdot \nabla(\rho_v + \rho_B) - \varepsilon \partial_{zz}(\rho_v + \rho_B) = \mathcal{H}_1 + \mathcal{H}_2$$

$$\partial_t(\rho_v - \rho_B) + (\omega_v + \omega_B) \cdot \nabla(\rho_v - \rho_B) - \varepsilon \partial_{zz}(\rho_v - \rho_B) = \mathcal{H}_1 - \mathcal{H}_2$$

This is heat equation with respect to z-direction, with zero boundary data on $z = 0$. We use heat kernel

$$G(t, y, z) = \frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(z-z')^2}{4t}} - e^{-\frac{(z+z')^2}{4t}} \right)$$

Using initial data ρ_0 , and source $(\mathcal{H}_1 + \mathcal{H}_2) - (\omega_v - \omega_B) \cdot \nabla(\rho_v + \rho_B)$, we get

$$(9.53) \quad \begin{aligned} \sqrt{\varepsilon} \partial_z(\rho_v + \rho_B)(t, y, z) &= \int_{-\infty}^0 \sqrt{\varepsilon} \partial_z G(t, z, z') \rho_0(y, z') dz' \\ &+ \int_0^t \int_{-\infty}^0 \sqrt{\varepsilon} \partial_z G(t - \tau, z, z') \{ \mathcal{H}_1 + \mathcal{H}_2 - (\omega_v - \omega_B) \cdot \nabla(\rho_v + \rho_B) \}(\tau, y, z') dz' d\tau \end{aligned}$$

Since G has a gaussian form we get

$$\begin{aligned} \sqrt{\varepsilon} \|\partial_z(\rho_v + \rho_B)(t)\|_{L^\infty} &\leq \sqrt{\varepsilon} \|\partial_z(\rho_v + \rho_B)(0)\|_{L^\infty} \\ &+ \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} (\|\mathcal{H}_1 + \mathcal{H}_2\|_{L^\infty} + \|(\omega_v - \omega_B) \cdot \nabla(\rho_v + \rho_B)\|_{L^\infty}) d\tau \end{aligned}$$

I) Using previous estimate for $\|\mathcal{H}_1\|_{1,\infty}, \|\mathcal{H}_2\|_{1,\infty}$

$$\begin{aligned} \|\mathcal{H}_1\|_{1,\infty} &\leq \Lambda_{\infty,m} (1 + \varepsilon \|\mathbf{S}^\varphi \mathcal{V}^m\| + \varepsilon \|S_n^v\|_{2,\infty} + \varepsilon \|v\|_{3,\infty}) \\ \|\mathcal{H}_2\|_{1,\infty} &\leq \Lambda_{\infty,m} (1 + \varepsilon \|S_n^B\|_{2,\infty} + \varepsilon \|B\|_{3,\infty}) \end{aligned}$$

We should control $\varepsilon \|S_n^v\|_{2,\infty}, \varepsilon \|v\|_{3,\infty}, \varepsilon \|S_n^B\|_{2,\infty}$, and $\varepsilon \|B\|_{3,\infty}$

$$\begin{aligned} \varepsilon \|S_n^v\|_{2,\infty} &\leq \varepsilon \|\nabla S_n^v\|_3^{\frac{1}{2}} \|S_n^v\|_4 \leq \Lambda_{\infty,m} \varepsilon \|\nabla S_n^v\|_{m-2}^{\frac{1}{2}} \\ \varepsilon \|S_n^B\|_{2,\infty} &\leq \varepsilon \|\nabla S_n^B\|_3^{\frac{1}{2}} \|S_n^B\|_4 \leq \Lambda_{\infty,m} \varepsilon \|\nabla S_n^B\|_{m-2}^{\frac{1}{2}} \\ \varepsilon \|v\|_{3,\infty} &\leq \varepsilon \|\nabla v\|_4^{\frac{1}{2}} \|v\|_5^{\frac{1}{2}} \leq \Lambda_{\infty,m} (1 + \|\nabla \mathcal{V}^m\|) \\ \varepsilon \|B\|_{3,\infty} &\leq \varepsilon \|\nabla B\|_4^{\frac{1}{2}} \|B\|_5^{\frac{1}{2}} \leq \Lambda_{\infty,m} (1 + \|\nabla \mathcal{B}^m\|) \end{aligned}$$

II)

$$\sqrt{\varepsilon} \|\partial_z \rho_v(0)\|_{L^\infty} \leq \Lambda_{\infty,m}(0), \quad \sqrt{\varepsilon} \|\partial_z \rho_B(0)\|_{L^\infty} \leq \Lambda_{\infty,m}(0)$$

III) Using the fact that ω is zero on the boundary, as we did, we give ∂_z to ω and tame the second term into conormal regularity not the ∂_z regularity, which is severe.

$$\|(\omega_v - \omega_B) \cdot \nabla(\rho_v + \rho_B)\|_{L^\infty} \leq \|\omega_v - \omega_B\|_{E^{1,\infty}} \|\rho_v + \rho_B\|_{1,\infty} \leq \Lambda_{\infty,m}$$

Hence, similar for couple equation,

$$(9.54) \quad \begin{aligned} \sqrt{\varepsilon} \|\partial_z(\rho_v + \rho_B)(t)\|_{L^\infty}, \sqrt{\varepsilon} \|\partial_z(\rho_v - \rho_B)(t)\|_{L^\infty} &\leq \Lambda_{\infty,m}(0) \\ &+ \int_0^t \frac{\Lambda_{\infty,m}}{\sqrt{t-\tau}} \left(1 + \varepsilon \|\nabla \mathcal{V}^m\|^{\frac{1}{2}} + \varepsilon \|\nabla \mathcal{B}^m\|^{\frac{1}{2}} + \varepsilon \|\nabla S_n^v\|_{m-2}^{\frac{1}{2}} + \varepsilon \|\nabla S_n^B\|_{m-2}^{\frac{1}{2}} \right) d\tau \end{aligned}$$

As we did before,

$$(9.55) \quad \begin{aligned} \sqrt{\varepsilon} \|\partial_z \rho_v(t)\|_{L^\infty}, \sqrt{\varepsilon} \|\partial_z \rho_B(t)\|_{L^\infty} &\leq \Lambda_{\infty,m}(0) \\ &+ \int_0^t \frac{\Lambda_{\infty,m}}{\sqrt{t-\tau}} \left(1 + \varepsilon \|\nabla \mathcal{V}^m\|^{\frac{1}{2}} + \varepsilon \|\nabla \mathcal{B}^m\|^{\frac{1}{2}} + \varepsilon \|\nabla S_n^v\|_{m-2}^{\frac{1}{2}} + \varepsilon \|\nabla S_n^B\|_{m-2}^{\frac{1}{2}} \right) d\tau \end{aligned}$$

Now we square this inequality. Main stuff is squaring the last term of this inequality. There are two cases. First, when we product two different terms we can use young's inequality to get the terms like $\varepsilon \int_0^t \|\nabla \mathcal{V}^m\| d\tau$, what we want. When a terms is squared, we should be careful, because if we use Holder's inequality for $L^2 - L^2$ then we may get terms like $\int_0^t \frac{\Lambda_{\infty,m}}{(t-\tau)^2} d\tau$. This is bad, since blow up near zero. So we use $L^4 - L^4 - L^2$ holder's inequality, to get, for example,

$$\left(\int_0^t \frac{\Lambda_{\infty,m}}{\sqrt{t-\tau}} \|\nabla S_n^v\|_{m-2}^{\frac{1}{2}} d\tau \right)^2 = \left(\int_0^t \frac{\Lambda_{\infty,m}}{(t-\tau)^{\frac{1}{8}}} \|\nabla S_n^v\|_{m-2}^{\frac{1}{2}} \frac{1}{(t-\tau)^{\frac{3}{8}}} d\tau \right)^2$$

$$\begin{aligned}
&\leq \left(\left(\int_0^t \frac{\Lambda_{\infty,m}^4}{\sqrt{t-\tau}} \right)^{\frac{1}{4}} \left(\int_0^t \|\nabla S_n^v\|_{m-2}^2 \right)^{\frac{1}{4}} \left(\int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \right)^{\frac{1}{2}} \right)^2 \\
&\leq \left(\int_0^t \frac{\Lambda_{\infty,m}^4}{\sqrt{t-\tau}} \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla S_n^v\|_{m-2}^2 \right)^{\frac{1}{2}} 4t^{\frac{1}{4}}
\end{aligned}$$

Now we can use Young's inequality to get the terms what we want. We skip other terms, since they are nearly similar. Finally we get (use $(a+b)^2 \leq 2(a^2+b^2)$),

$$\begin{aligned}
(9.56) \quad &\varepsilon \|\partial_z \rho_v\|_{L^\infty}^2, \quad \varepsilon \|\partial_z \rho_B\|_{L^\infty}^2 \leq \Lambda_{\infty,m}(0) \\
&+ 2 \int_0^t (\varepsilon \|\nabla \mathcal{V}^m\|^2 + \varepsilon \|\nabla \mathcal{B}^m\|^2 + \varepsilon \|\nabla S_n^v\|_{m-2}^2 + \varepsilon \|\nabla S_n^B\|_{m-2}^2) + (1+16\sqrt{t}) \int_0^t \frac{\Lambda_{\infty,m}}{\sqrt{t-\tau}} d\tau
\end{aligned}$$

□

9.4. $\int_0^t \sqrt{\varepsilon} \|\nabla^2 v\|_{1,\infty}, \int_0^t \sqrt{\varepsilon} \|\nabla^2 B\|_{1,\infty}$ **estimate.** We need estimates of $\int_0^t \sqrt{\varepsilon} \|\nabla^2 v\|_{1,\infty}, \int_0^t \sqrt{\varepsilon} \|\nabla^2 B\|_{1,\infty}$ later.

Lemma 9.8. *Let $m \geq 6$ and $\sup_{[0,T]} \Lambda_{m,\infty}(t) \leq M$. Then*

$$(9.57) \quad \sqrt{\varepsilon} \int_0^t \|\partial_{zz} v\|_{1,\infty} \leq \Lambda(M)(1+16\sqrt{t})\sqrt{t} \left(1 + \varepsilon \int_0^T (\|\nabla \mathcal{V}^m\|^2 + \|\nabla \mathcal{B}^m\|^2 + \|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) \right)$$

This is same for $\int_0^t \sqrt{\varepsilon} \|\partial_{zz} v\|_{1,\infty}$.

Proof. It is similar with proposition 9.7. One more conormal derivative order changes nearly nothing, since our m is sufficient. Integration for t changes $\frac{1}{\sqrt{t-\tau}}$ into \sqrt{t} . □

Proposition 9.9. *Under the same assumption, we get the following.*

$$(9.58) \quad \sqrt{\varepsilon} \int_0^t \|\nabla^2 v\|_{1,\infty} \leq \Lambda(M)(1+t)^2 \left(1 + \varepsilon \int_0^t (\|\nabla \mathcal{V}^m\|^2 + \|\nabla \mathcal{B}^m\|^2 + \|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) \right)$$

This is also valid for B .

Proof. First, estimate of $\sqrt{\varepsilon} \int_0^t \|\nabla v\|_{2,\infty}$ is easy, because we know

$$\|\nabla v\|_{2,\infty} \leq \Lambda_{m,\infty}(\|\nabla S_n^v\|_{2,\infty} + \|v\|_{3,\infty})$$

We get easily the following.

$$\sqrt{\varepsilon} \int_0^t \|\nabla v\|_{2,\infty} \leq \int_0^t (\Lambda_{m,\infty} + \varepsilon \|\nabla \mathcal{V}^m\|^2 + \varepsilon \|\nabla S_n^v\|_{m-2}^2)$$

Next, with help of Lemma 9.8, we have everything we need to finish the proof. We skip the detail. □

10. VORTICITY ESTIMATE

We could not get $\|\partial_z v\|_{L_t^\infty H_{co}^{m-1}}$ estimate in previous sections. ($m-2$ estimate was optimal.) But if we weaken L_t^∞ to some weak $L_t^{\alpha \geq 2}$ (which is also sufficient) we may get the $m-1$ regularity. Now we write vorticity of v, B as ω_v, ω_B . First we define (same for B)

$$(10.1) \quad \omega_v = \nabla^\varphi \times v, \quad \omega_v = (\nabla \times u)(t, \Phi)$$

Note that (also for B)

$$\|Z^{m-1} \partial_z v\| \leq \Lambda_{m,\infty}(\|v\|_m + |h|_{m-\frac{1}{2}} + \|\omega_v\|_{m-1})$$

which implies that we are suffice to control $\|\omega_{v,B}\|_{m-1}$ to control $\|\partial_z v\|_{m-1}$. Applying $\nabla^\varphi \times$ kill pressure so Navier-Stokes and Faraday get nearly same structure.

$$\begin{aligned}
(10.2) \quad &\partial_t^\varphi \omega_v + (v \cdot \nabla^\varphi) \omega_v - (B \cdot \nabla^\varphi) \omega_B - (\omega_v \cdot \nabla^\varphi) v + (\omega_B \cdot \nabla^\varphi) B = \varepsilon \Delta^\varphi \omega_v \\
&\partial_t^\varphi \omega_B + (v \cdot \nabla^\varphi) \omega_B - (B \cdot \nabla^\varphi) \omega_v - (\omega_v \cdot \nabla^\varphi) B + (\omega_B \cdot \nabla^\varphi) v = \varepsilon \Delta^\varphi \omega_B
\end{aligned}$$

Situation is quiet different from S_n case. The reason we took S_n instead of $\partial_z v$ is that it is equivalent to $\partial_z v$ and moreover it is zero boundary condition, (since, $\mathbf{S}^\varphi \mathbf{n}^b$ has only normal component by compatibility

condition), so it is very easy to get estimate. It is more complicate than $\mathbf{S}^\varphi v \mathbf{n}$, but vanishing on the boundary is a big advantage. But main difficulty of vorticity is that

$$\omega \times \mathbf{n} = \Pi(\omega \times \mathbf{n}) \neq 0 \text{ on } \partial S$$

That means $\omega \times \mathbf{n}$ has no advantage to use than use of ω . Thus we just use $\|\omega\|_{m-1}$ directly. Applying Z^α gives, ($|\alpha| \leq m-1$)

$$(10.3) \quad \partial_t^\varphi Z^\alpha \omega_v + (v \cdot \nabla^\varphi) Z^\alpha \omega_v - (B \cdot \nabla^\varphi) Z^\alpha \omega_B - \varepsilon \Delta^\varphi Z^\alpha \omega_v = F$$

where

$$F = Z^\alpha(\omega_v \cdot \nabla^\varphi v) - Z^\alpha(\omega_B \cdot \nabla^\varphi B) + \mathcal{C}_S$$

where

$$\mathcal{C}_S = \mathcal{C}_S^1 + \mathcal{C}_S^2 + \mathcal{C}_S^3$$

with

$$(10.4) \quad \begin{aligned} \mathcal{C}_S^1 &= [Z^\alpha v_y] \cdot \nabla_y \omega_v + [Z^\alpha, V_z] \partial_z \omega_v \doteq \mathcal{C}_{S_y}^1 + \mathcal{C}_{S_z}^1 \\ \mathcal{C}_S^2 &= \varepsilon [Z^\alpha, \Delta^\varphi] \omega_v \\ \mathcal{C}_S^3 &= -[Z^\alpha B_y] \cdot \nabla_y \omega_B + [Z^\alpha, \frac{B \cdot N}{\partial_z \varphi}] \partial_z \omega_B \doteq \mathcal{C}_{S_y}^3 + \mathcal{C}_{S_z}^3 \\ \partial_t^\varphi Z^\alpha \omega_B + (v \cdot \nabla^\varphi) Z^\alpha \omega_B - (B \cdot \nabla^\varphi) Z^\alpha \omega_v - \varepsilon \Delta^\varphi Z^\alpha \omega_B &= E \end{aligned}$$

where

$$E = Z^\alpha(\omega_v \cdot \nabla^\varphi B) - Z^\alpha(\omega_B \cdot \nabla^\varphi v) + \bar{\mathcal{C}}_S$$

where

$$\bar{\mathcal{C}}_S = \bar{\mathcal{C}}_S^1 + \bar{\mathcal{C}}_S^2 + \bar{\mathcal{C}}_S^3$$

with

$$\begin{aligned} \bar{\mathcal{C}}_S^1 &= [Z^\alpha v_y] \cdot \nabla_y \omega_B + [Z^\alpha, V_z] \partial_z \omega_B \doteq \bar{\mathcal{C}}_{S_y}^1 + \bar{\mathcal{C}}_{S_z}^1 \\ \bar{\mathcal{C}}_S^2 &= \varepsilon [Z^\alpha, \Delta^\varphi] \omega_B \\ \bar{\mathcal{C}}_S^3 &= -[Z^\alpha B_y] \cdot \nabla_y \omega_v + [Z^\alpha, \frac{B \cdot \mathbf{N}}{\partial_z \varphi}] \partial_z \omega_v \doteq \bar{\mathcal{C}}_{S_y}^3 + \bar{\mathcal{C}}_{S_z}^3 \end{aligned}$$

And for boundary data we have, by lemma 5.5 in [1]

$$(10.5) \quad |(Z^\alpha \omega_v)^b| \leq \Lambda_{\infty,6} (|v|_m^b + |h|_m)$$

$$(10.6) \quad |(Z^\alpha \omega_B)^b| \leq \Lambda_{\infty,6} (|B|_m^b + |h|_m)$$

Since Lemma 5.5 use divergence free condition and compatibility condition, which is also true for B . In fact B is stronger than v , and just suffice to estimate normal compo, since tangential compo is zero by no slip B . Then using trace theorem,

$$\begin{aligned} |(Z^\alpha \omega_v)^b| &\leq \Lambda_{\infty,6} \left(\|\nabla \mathcal{V}^m\|^{\frac{1}{2}} \|\mathcal{V}^m\|^{\frac{1}{2}} + \|\mathcal{V}^m\| + |h|_m \right) \\ |(Z^\alpha \omega_B)^b| &\leq \Lambda_{\infty,6} \left(\|\nabla \mathcal{B}^m\|^{\frac{1}{2}} \|\mathcal{B}^m\|^{\frac{1}{2}} + \|\mathcal{B}^m\| + |h|_m \right) \end{aligned}$$

then the only way to control this boundary terms is,

$$\begin{aligned} \sqrt{\varepsilon} \int_0^t |(Z^\alpha \omega_v)^b|^2 &\leq \varepsilon \int_0^t \|\nabla \mathcal{V}^m\|^2 + \int_0^t \Lambda_{\infty,6} (\|\mathcal{V}^m\|^2 + |h|_m^2) \\ \sqrt{\varepsilon} \int_0^t |(Z^\alpha \omega_B)^b|^2 &\leq \varepsilon \int_0^t \|\nabla \mathcal{B}^m\|^2 + \int_0^t \Lambda_{\infty,6} (\|\mathcal{B}^m\|^2 + |h|_m^2) \end{aligned}$$

It will be useful to keep the term before using young's inequality

$$(10.7) \quad \begin{aligned} \sqrt{\varepsilon} \int_0^t |(Z^\alpha \omega_v)^b|^2 &\leq \Lambda_{\infty,6} \sqrt{\varepsilon} (\|\nabla \mathcal{V}^m\| \|\mathcal{V}^m\| + \|\mathcal{V}^m\|^2 + |h|_m^2) \\ \sqrt{\varepsilon} \int_0^t |(Z^\alpha \omega_B)^b|^2 &\leq \Lambda_{\infty,6} \sqrt{\varepsilon} (\|\nabla \mathcal{B}^m\| \|\mathcal{B}^m\| + \|\mathcal{B}^m\|^2 + |h|_m^2) \end{aligned}$$

Now as like in [1], we split

$$Z^\alpha \omega_v = \omega_{v,h}^\alpha + \omega_{v,nh}^\alpha, \quad Z^\alpha \omega_B = \omega_{B,h}^\alpha + \omega_{B,nh}^\alpha$$

I) $\omega_{v,nh}^\alpha, \omega_{B,nh}^\alpha$ solves, nonhomogeneous

$$(10.8) \quad \partial_t^\varphi \omega_{v,nh}^\alpha + (v \cdot \nabla^\varphi) \omega_{v,nh}^\alpha - (B \cdot \nabla^\varphi) \omega_{B,nh}^\alpha - \varepsilon \Delta^\varphi \omega_{v,nh}^\alpha = F$$

with initial and zero-boundary condition,

$$(\omega_{v,nh}^\alpha)^b = 0, \quad (\omega_{v,nh}^\alpha)_{t=0} = \omega_v(0)$$

and

$$(10.9) \quad \partial_t^\varphi \omega_{B,nh}^\alpha + (v \cdot \nabla^\varphi) \omega_{B,nh}^\alpha - (B \cdot \nabla^\varphi) \omega_{v,nh}^\alpha - \varepsilon \Delta^\varphi \omega_{B,nh}^\alpha = E$$

with initial and zero-boundary condition,

$$(\omega_{B,nh}^\alpha)^b = 0, \quad (\omega_{B,nh}^\alpha)_{t=0} = \omega_B(0)$$

II) $\omega_{v,h}^\alpha, \omega_{B,h}^\alpha$ solves, homogeneous

$$(10.10) \quad \partial_t^\varphi \omega_{v,h}^\alpha + (v \cdot \nabla^\varphi) \omega_{v,h}^\alpha - (B \cdot \nabla^\varphi) \omega_{B,h}^\alpha - \varepsilon \Delta^\varphi \omega_{v,h}^\alpha = 0$$

with initial and -boundary condition,

$$(\omega_{v,h}^\alpha)^b = (Z^\alpha \omega_v)^b, \quad (\omega_{v,h}^\alpha)_{t=0} = 0$$

and

$$(10.11) \quad \partial_t^\varphi \omega_{B,h}^\alpha + (v \cdot \nabla^\varphi) \omega_{B,h}^\alpha - (B \cdot \nabla^\varphi) \omega_{v,h}^\alpha - \varepsilon \Delta^\varphi \omega_{B,h}^\alpha = 0$$

with initial and -boundary condition,

$$(\omega_{B,h}^\alpha)^b = (Z^\alpha \omega_B)^b, \quad (\omega_{B,h}^\alpha)_{t=0} = 0$$

$\omega_{v,h}^\alpha, \omega_{B,h}^\alpha$ can be separated again as follow.

$$\omega_{v,h}^\alpha = \omega_{v,h}^{\alpha,1} + \omega_{v,h}^{\alpha,2}, \quad \omega_{B,h}^\alpha = \omega_{B,h}^{\alpha,1} + \omega_{B,h}^{\alpha,2}$$

II-1) $\omega_{v,h}^{\alpha,1}, \omega_{B,h}^{\alpha,1}$ solves

$$(10.12) \quad \partial_t^\varphi \omega_{v,h}^{\alpha,1} + (v \cdot \nabla^\varphi) \omega_{v,h}^{\alpha,1} - \varepsilon \Delta^\varphi \omega_{v,h}^{\alpha,1} = 0$$

with initial and zero-boundary condition,

$$(\omega_{v,h}^{\alpha,1})^b = (Z^\alpha \omega_v)^b, \quad (\omega_{v,h}^{\alpha,1})_{t=0} = 0$$

and

$$(10.13) \quad \partial_t^\varphi \omega_{B,h}^{\alpha,1} + (v \cdot \nabla^\varphi) \omega_{B,h}^{\alpha,1} - \varepsilon \Delta^\varphi \omega_{B,h}^{\alpha,1} = 0$$

with initial and zero-boundary condition,

$$(\omega_{B,h}^{\alpha,1})^b = (Z^\alpha \omega_B)^b, \quad (\omega_{B,h}^{\alpha,1})_{t=0} = 0$$

II-2) $\omega_{v,h}^{\alpha,2}, \omega_{B,h}^{\alpha,2}$ solves

$$(10.14) \quad \partial_t^\varphi \omega_{v,h}^{\alpha,2} + (v \cdot \nabla^\varphi) \omega_{v,h}^{\alpha,2} - (B \cdot \nabla^\varphi) \omega_{B,h}^{\alpha,2} - \varepsilon \Delta^\varphi \omega_{v,h}^{\alpha,2} = 0$$

with initial and zero-boundary condition,

$$(\omega_{v,h}^{\alpha,2})^b = 0, \quad (\omega_{v,h}^{\alpha,2})_{t=0} = 0$$

and

$$(10.15) \quad \partial_t^\varphi \omega_{B,h}^{\alpha,2} + (v \cdot \nabla^\varphi) \omega_{B,h}^{\alpha,2} - (B \cdot \nabla^\varphi) \omega_{v,h}^{\alpha,2} - \varepsilon \Delta^\varphi \omega_{B,h}^{\alpha,2} = 0$$

with initial and zero-boundary condition,

$$(\omega_{B,h}^{\alpha,2})^b = 0, \quad (\omega_{B,h}^{\alpha,2})_{t=0} = 0$$

We state the energy estimate for these two vorticity terms. First let's define,

$$(10.16) \quad \begin{aligned} \|\omega_{v,nh}^{m-1}\|^2 &\doteq \sum_{|\alpha| \leq m-1} \|\omega_{v,nh}^\alpha\|^2, \quad \int_0^t \|\nabla \omega_{v,nh}^{m-1}\|^2 \doteq \int_0^t \sum_{|\alpha| \leq m-1} \|\nabla \omega_{v,nh}^\alpha\|^2 \\ \|\omega_{B,nh}^{m-1}\|^2 &\doteq \sum_{|\alpha| \leq m-1} \|\omega_{B,nh}^\alpha\|^2, \quad \int_0^t \|\nabla \omega_{B,nh}^{m-1}\|^2 \doteq \int_0^t \sum_{|\alpha| \leq m-1} \|\nabla \omega_{B,nh}^\alpha\|^2 \end{aligned}$$

$$\begin{aligned}
(10.17) \quad & \|\omega_{v,h}^{m-1}\|^2 \doteq \sum_{|\alpha| \leq m-1} \|\omega_{v,h}^\alpha\|^2 = \sum_{|\alpha| \leq m-1} \left(\|\omega_{v,h}^{\alpha,1}\|^2 + \|\omega_{v,h}^{\alpha,2}\|^2 \right) \\
& \int_0^t \|\nabla \omega_{v,h}^{m-1}\|^2 \doteq \int_0^t \sum_{|\alpha| \leq m-1} \|\nabla \omega_{v,h}^\alpha\|^2 = \int_0^t \sum_{|\alpha| \leq m-1} \left(\|\nabla \omega_{v,h}^{\alpha,1}\|^2 + \|\nabla \omega_{v,h}^{\alpha,2}\|^2 \right) \\
& \|\omega_{B,h}^{m-1}\|^2 \doteq \sum_{|\alpha| \leq m-1} \|\omega_{B,h}^\alpha\|^2 = \sum_{|\alpha| \leq m-1} \left(\|\omega_{B,h}^{\alpha,1}\|^2 + \|\omega_{B,h}^{\alpha,2}\|^2 \right) \\
& \int_0^t \|\nabla \omega_{B,h}^{m-1}\|^2 \doteq \int_0^t \sum_{|\alpha| \leq m-1} \|\nabla \omega_{B,h}^\alpha\|^2 = \int_0^t \sum_{|\alpha| \leq m-1} \left(\|\nabla \omega_{B,h}^{\alpha,1}\|^2 + \|\nabla \omega_{B,h}^{\alpha,2}\|^2 \right)
\end{aligned}$$

Note that to control $\|\omega_{v,B}\|_{m-1}^2$, for both v, B ,

$$\begin{aligned}
(10.18) \quad & \|\omega\|_{m-1}^2 = \sum_{|\alpha| \leq m-1} \|Z^\alpha \omega\|^2 = \sum_{|\alpha| \leq m-1} \|\omega_h^\alpha + \omega_{nh}^\alpha\|^2 \leq 2\|\omega_{nh}^{m-1}\|^2 + 2\|\omega_h^{m-1}\|^2 \\
& \leq 2 \sum_{|\alpha| \leq m-1} \|\omega_{nh}^\alpha\|^2 + 2 \sum_{|\alpha| \leq m-1} \|\omega_h^{\alpha,1}\|^2 + 2 \sum_{|\alpha| \leq m-1} \|\omega_h^{\alpha,2}\|^2
\end{aligned}$$

we are suffice to control $\|Z^\alpha \omega_h^1\|^2$, $\|Z^\alpha \omega_h^2\|^2$, $\|Z^\alpha \omega_{nh}\|^2$ and sum for all α .

10.1. Non-homogeneous estimate. We estimate $\omega_{v,nh}^\alpha, \omega_{B,nh}^\alpha$

Proposition 10.1. *We have the following vorticity estimates for $\omega_{v,nh}^\alpha$.*

$$\begin{aligned}
(10.19) \quad & \|\omega_{v,nh}^{m-1}\|^2 + 2\varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{v,nh}^{m-1}|^2 dV_t ds - 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \omega_{B,nh}^{m-1} \cdot \omega_{v,nh}^{m-1} dV_t ds \\
& \leq \Lambda_0 \|\omega_{v,nh}^{m-1}(0)\|^2 + \int_0^t \Lambda_{\infty,m} \left(\|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\omega_v\|_{m-1}^2 + \|\omega_B\|_{m-1}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) \\
& \quad + \varepsilon \int_0^t \Lambda_{\infty,m} (\|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) ds
\end{aligned}$$

Proof. Using equation for $\omega_{v,nh}^\alpha$, (with dirichlet boundary condition) we get L^2 type energy estimate.

$$(10.20) \quad \frac{1}{2} \frac{d}{dt} \int_S |\omega_{v,nh}^\alpha|^2 dV_t + \varepsilon \int_S |\nabla^\varphi \omega_{v,nh}^\alpha|^2 dV_t - \int_S (B \cdot \nabla^\varphi) \omega_{B,nh}^\alpha \cdot \omega_{v,nh}^\alpha dV_t = \int_S F \cdot \omega_{v,nh}^\alpha dV_t$$

I) $\|Z^\alpha(\omega_v \cdot \nabla^\varphi v)\|, \|Z^\alpha(\omega_B \cdot \nabla^\varphi B)\|$ estimate.

Simply we get

$$\begin{aligned}
& \|Z^\alpha(\omega_v \cdot \nabla^\varphi v)\| \leq \Lambda_{\infty,m} \left(\|\omega_v\|_{m-1} + \|v\|_m + |h|_{m-\frac{1}{2}} \right) \\
& \|Z^\alpha(\omega_B \cdot \nabla^\varphi B)\| \leq \Lambda_{\infty,m} \left(\|\omega_B\|_{m-1} + \|B\|_m + |h|_{m-\frac{1}{2}} \right)
\end{aligned}$$

II) $\left| \int_S \mathcal{C}_S^2 \cdot \omega_{v,nh}^\alpha dV_t \right|$ estimate.

As like in $S_n^{v,B}$ estimate,

$$\left| \int_S \mathcal{C}_S^2 \cdot \omega_{v,nh}^\alpha dV_t \right| \leq \Lambda_0 \left(\sqrt{\varepsilon} \|\nabla^\varphi \omega_{v,nh}^\alpha\| + \|\omega_v\|_{m-1} \right) \left(\sqrt{\varepsilon} \|\nabla \omega_v\|_{m-2} + \|\omega_v\|_{m-1} + \Lambda_{\infty,m} \left(|h|_{m-\frac{1}{2}} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right) \right)$$

III) $\|\mathcal{C}_{S_y}^1\|, \|\mathcal{C}_{S_y}^3\|$ estimate.

This is also similar to previous $S_n^{v,B}$ estimate,

$$\begin{aligned}
& \|\mathcal{C}_{S_y}^1\| \leq \Lambda_{\infty,m} (\|\omega_v\|_{m-1} + \|v\|_m + |h|_m) \\
& \|\mathcal{C}_{S_y}^3\| \leq \Lambda_{\infty,m} (\|\omega_B\|_{m-1} + \|B\|_m + |h|_m)
\end{aligned}$$

IV) $\|\mathcal{C}_{S_z}^1\|, \|\mathcal{C}_{S_z}^3\|$ estimate.

This is the main part of proof. In $S_n^{v,B}$ section, $|\alpha| = m-2$ was optimal, since $|h|_{m-\frac{1}{2}}$ and $\|v\|_{E^m}$. Note that in pressure estimate, if we use $\nabla^\varphi \cdot (v \cdot \nabla^\varphi v) = \nabla^\varphi v : (\nabla^\varphi v)^T$ then we could get 1 regularity for v . Nevertheless $m-1$ is not available because of the worst h regularity term which come from $\|\nabla q^E\|$ in F_S and C_{S_z} . But now, since we consider vorticity, $\|\nabla q^E\|$ does not appear, so only the problem is $|h|_{m-\frac{1}{2}}$ in

C_{S_z} . What we show here is that we can get $\frac{1}{2}$ regularity of h in fact. (We didn't have to do in S_n section, since pressure generate $|h|_{m-\frac{1}{2}}$.)

Now to estimate $C_{S_z}^1$, we should estimate the terms like (with $|\gamma| + |\beta| \leq m-1$, $|\gamma| \leq m-2$)

$$\|Z^\beta V_z \partial_z Z^\gamma \omega_v\|$$

We write this as

$$c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1-z}{z} V_z \right) Z_3 Z^{\tilde{\gamma}} \omega_v$$

with $|\tilde{\gamma}| + |\tilde{\beta}| \leq m-1$, $|\tilde{\gamma}| \leq m-2$. Then using Lemma 8.4 in [1],

$$\leq \Lambda_{\infty, m} \left(\|\omega_v\|_{m-1} + |h|_{m-\frac{1}{2}} + \left\| \frac{1-z}{z} Z(v \cdot \mathbf{N} - \partial_z \varphi) \right\|_{m-2} \right)$$

Last term is the main stuff.

$$\left\| \frac{1-z}{z} Z(v \cdot \mathbf{N} - \partial_z \varphi) \right\|_{m-2} \leq \Lambda_{\infty, m} \left(\|v\|_{E^m} + \sum_{|\alpha| \leq m-1} \|v \cdot \partial_z Z^\alpha \mathbf{N} - \partial_z Z^\alpha \partial_t \varphi\|_{L^2} \right)$$

If we brutally estimate this, then we get

$$\|\partial_z Z^\alpha \mathbf{N}\| \sim |\nabla \varphi|_{m-1+1} \sim |\varphi|_{m+1} \sim |h|_{m+\frac{1}{2}}$$

so we loose $\frac{1}{2}$ regularity. We treat this carefully. First, we should see that Z_3 does not lose regularity of h . From definition of $\varphi(t, y, z) = Az + \eta(t, y, z)$,

$$|Z_3 \hat{\eta}| = \left| \frac{z}{1-z} \partial_z \hat{\eta}(\xi, z) \right| = \left| \frac{z}{1-z} \partial_z \left(\chi(\xi z) \hat{h}(\xi) \right) \right| \leq \left| \frac{z}{1-z} \xi \cdot \nabla \chi(\xi z) \right| \leq |\chi_2(\xi z) \hat{h}|$$

where $(\chi_2$ is 1 on $B_2(0)$, which is bigger support than χ)

This is because, $\nabla \chi$ has support on annular domain, $B_2(0) - B_1(0)$. Above regularity means applying Z_3 does not reduce 1 regularity. So, when $\alpha_3 \neq 0$ is not harmful, which means,

$$\|v \cdot \partial_z Z^\alpha \mathbf{N} - \partial_z Z^\alpha \partial_t \varphi\|_{L^2} \leq \Lambda_{\infty, m} \left(|h|_{m-\frac{1}{2}} + |\partial_t h|_{m-\frac{3}{2}} \right) \leq \Lambda_{\infty, m} \left(|h|_{m-\frac{1}{2}} + \|v\|_{E^m} \right), \quad \alpha_3 \neq 0$$

When $\alpha_3 = 0$,

$$v \cdot \partial_z Z^\alpha \mathbf{N} - \partial_z Z^\alpha \partial_t \eta = -v_1 \partial_z (\psi_z *_y \partial_1 Z^\alpha h) - v_2 \partial_z (\psi_z *_y \partial_2 Z^\alpha h) - \partial_z (\psi_z *_y \partial_t Z^\alpha h) \doteq \mathcal{T}_\alpha$$

where

$$\hat{\psi}_z = \chi(\xi z)$$

so by inverse fourier transformation with respect to horizontal variable,

$$\psi_z(y) = \frac{1}{z^2} \check{\chi} \left(\frac{y}{z} \right)$$

Note that χ is in Schwartz class so $\check{\chi}$ is also in Schwartz class. Moreover, when $|z| \leq 1$,

$$(10.21) \quad Z^\alpha \psi_z(z) = \mathbf{D}^\alpha \psi_z(z) = -\frac{2}{z^3} \check{\chi} \left(\frac{y}{z} \right) + \frac{1}{z^2} (\nabla \check{\chi}) \left(\frac{y}{z} \right) \cdot y \leq \frac{2}{z^3} (1 + |y|) \zeta \left(\frac{y}{z} \right)$$

For some function ζ with compact support near origin. This is because $\nabla \check{\chi}$ is also in Schwartz class.

i) When $z \leq -1$,

$$\begin{aligned} \|\mathcal{T}_\alpha\|_{L^2(S \cap |z| \geq 1)} &= \|-v_1 \partial_z (\mathbf{D}^\alpha \psi_z *_y \partial_1 h) - v_2 \partial_z (\mathbf{D}^\alpha \psi_z *_y \partial_2 h) - \partial_z (\mathbf{D}^\alpha \psi_z *_y \partial_t h)\| \\ &\leq \Lambda_{\infty, m} \left(\left\| \partial_z \left(\frac{1}{z^{m-1}} (\mathbf{D}^\alpha \psi_z) *_y \nabla h \right) \right\|_{L^2(S \cap |z| \geq 1)} + \left\| \partial_z \left(\frac{1}{z^{m-1}} (\mathbf{D}^\alpha \psi_z) *_y \partial_t h \right) \right\|_{L^2(S \cap |z| \geq 1)} \right) \end{aligned}$$

Since $\mathbf{D}^\alpha \psi_z$ is also similar to ψ_z , we get

$$(10.22) \quad \|\mathcal{T}_\alpha\|_{L^2(S \cap |z| \geq 1)} \leq \Lambda_{\infty, m} (\|v\|_{E^1} + |h|_1)$$

ii) When $z \geq -1$

$$\begin{aligned} \mathcal{T}_\alpha &= -v_1 \partial_z (\psi_z *_y \partial_1 Z^\alpha h) - v_2 \partial_z (\psi_z *_y \partial_2 Z^\alpha h) - \partial_z (\psi_z *_y \partial_t Z^\alpha h) \\ &= -v_1^b \partial_z (\psi_z *_y \partial_1 Z^\alpha h) - v_2^b \partial_z (\psi_z *_y \partial_2 Z^\alpha h) - \partial_z (\psi_z *_y \partial_t Z^\alpha h) + \mathcal{R} \end{aligned}$$

where

$$\mathcal{R} = (v_1^b - v_1) \partial_z (\psi_z *_y \partial_1 Z^\alpha h) + (v_2^b - v_2) \partial_z (\psi_z *_y \partial_2 Z^\alpha h)$$

Using taylor expansion with respect to z , we get

$$|\mathcal{R}| \leq \Lambda_{\infty, m} |z| |\partial_z (\psi_z *_y Z^\alpha \nabla h)| \leq |Z_3 (\psi_z *_y Z^\alpha \nabla h)|$$

Note that Z_3 do nothing about regularity to ψ_z , so

$$(10.23) \quad \|\mathcal{R}\| \leq \Lambda_{\infty, m} |h|_{m-\frac{1}{2}}$$

For first part in \mathcal{T}_α ,

$$\begin{aligned} & -v_1^b \partial_z (\psi_z *_y \partial_1 Z^\alpha h) - v_2^b \partial_z (\psi_z *_y \partial_2 Z^\alpha h) - \partial_z (\psi_z *_y \partial_t Z^\alpha h) \\ & = \partial_z (\psi_z *_y (-v_1^b \partial_1 Z^\alpha h - v_2^b \partial_2 Z^\alpha h - \partial_t Z^\alpha h)) \\ & + \partial_z \int_S ((v_1(t, y', 0) - v_1(t, y, 0)) \psi_z(y - y') \partial_1 Z^\alpha h(t, y') + (v_2(t, y', 0) - v_2(t, y, 0)) \psi_z(y - y') \partial_2 Z^\alpha h(t, y')) dy \end{aligned}$$

By taylor expansion,

$$|(v_i^b(t, y', 0) - v_i^b(t, y, 0)) \partial_z \psi_z(y - y')| \leq |\nabla_y v_i^b|_{L^\infty} \left| (y - y') \frac{2}{z^3} \partial_z \psi\left(\frac{y - y'}{z}\right) \right|$$

So, for $\forall z \in (0, 1]$,

$$\begin{aligned} \sup_{z \in (0, 1]} & \leq \left\| \partial_z \int_{\partial S} (v_i(t, y', 0) - v_i(t, y, 0)) \psi_z(y - y') \partial_i Z^\alpha h(t, y') \right\| \\ & \leq \|\nabla_y v_i^b\|_{L^\infty} \sup_{z \in (0, 1]} \int_{\partial S} \left| \frac{y - y'}{z^3} \right|^2 \zeta^2\left(\frac{y - y'}{z}\right) z^2 d\left(\frac{y - y'}{z}\right) \\ & \leq \|\nabla_y v_i^b\|_{L^\infty} \sup_{z \in (0, 1]} \int_{\partial S} y^2 \zeta^2(y) dy \leq \Lambda_{\infty, m} \end{aligned}$$

For the first term of \mathcal{T}_α ,

$$\| -v_1^b \partial_1 Z^\alpha h - v_2^b \partial_2 Z^\alpha h - \partial_t Z^\alpha h \|_{H^{\frac{1}{2}}} = \| \mathcal{C}^\alpha(h) - (\mathcal{V}^\alpha)^b - v_3^b \|_{H^{\frac{1}{2}}} \leq \Lambda_{\infty, m} (\|v\|_{E^m} + |h|_{m-\frac{1}{2}})$$

thus

$$\| \partial_z (\psi_z *_y (-v_1^b \partial_1 Z^\alpha h - v_2^b \partial_2 Z^\alpha h - \partial_t Z^\alpha h)) \| \leq \Lambda_{\infty, m} (\|v\|_{E^m} + |h|_{m-\frac{1}{2}})$$

Finally we get

$$\|\mathcal{T}_\alpha\|_{L^2(S \cap \{|z| \neq 1\})} \leq \Lambda_{\infty, m} (\|v\|_{E^m} + |h|_m)$$

Considering i) and ii), and $\alpha_3 \neq 0$,

$$(10.24) \quad \|\mathcal{C}_{SZ}^1\| \leq \Lambda_{\infty, m} (\|\omega_v\|_{m-1} + \|v\|_{E^m} + |h|_m)$$

For $\|\mathcal{C}_{S_z}^3\|$, we can do similar thing for B , since B is zero on the boundary, so control is better.

$$(10.25) \quad \|\mathcal{C}_{SZ}^3\| \leq \Lambda_{\infty, m} (\|\omega_B\|_{m-1} + \|B\|_{E^m} + |h|_m)$$

Considering I) IV),

$$\begin{aligned} & \|\omega_{v, nh}^\alpha\|^2 + 2\varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{v, nh}^\alpha|^2 dV_t ds - 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \omega_{B, nh}^\alpha \cdot \omega_{v, nh}^\alpha dV_t ds \\ & \leq \Lambda_0 \|\omega_{v, nh}^\alpha(0)\|^2 + \int_0^t \Lambda_{\infty, m} (\|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\omega_v\|_{m-1}^2 + \|\omega_B\|_{m-1}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2) \\ & \quad + \Lambda_0 \varepsilon \int_0^t (\|\nabla \omega_v\|_{m-2}^2 + \|\nabla \omega_B\|_{m-2}^2) ds \end{aligned}$$

For the last term,

$$\sqrt{\varepsilon} \|\nabla \omega_v\|_{m-2} \leq \Lambda_{\infty, m} (\sqrt{\varepsilon} \|\partial_{zz} v\|_{m-2} + \sqrt{\varepsilon} \|\partial_z v\|_{m-1}) + \Lambda_{\infty, m} |h|_{m-\frac{1}{2}}$$

This gives, (by summing for all indices)

$$(10.26) \quad \|\omega_{v, nh}^{m-1}\|^2 + 2\varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{v, nh}^{m-1}|^2 dV_t ds - 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \omega_{B, nh}^{m-1} \cdot \omega_{v, nh}^{m-1} dV_t ds$$

$$\begin{aligned} &\leq \Lambda_0 \|\omega_{v,nh}^{m-1}(0)\|^2 + \int_0^t \Lambda_{\infty,m} \left(\|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\omega_v\|_{m-1}^2 + \|\omega_B\|_{m-1}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) \\ &\quad + \varepsilon \int_0^t \Lambda_{\infty,m} (\|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) ds \end{aligned}$$

□

For $\omega_{B,nh}^\alpha$, similarly,

Proposition 10.2. *We have the following vorticity estimates for $\omega_{B,nh}^\alpha$.*

$$\begin{aligned} (10.27) \quad &\|\omega_{B,nh}^{m-1}\|^2 + 2\varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{B,nh}^{m-1}|^2 dV_t ds + 2 \int_0^t \int_S (B \cdot \nabla^\varphi) \omega_{B,nh}^{m-1} \cdot \omega_{v,nh}^{m-1} dV_t ds \\ &\leq \Lambda_0 \|\omega_{B,nh}^{m-1}(0)\|^2 + \int_0^t \Lambda_{\infty,m} \left(\|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\omega_v\|_{m-1}^2 + \|\omega_B\|_{m-1}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) \\ &\quad + \varepsilon \int_0^t \Lambda_{\infty,m} (\|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) ds \end{aligned}$$

Proof. Using equation for $\omega_{B,nh}^\alpha$, (with dirichlet boundary condition) we get L^2 type energy estimate.

$$(10.28) \quad \frac{1}{2} \frac{d}{dt} \int_S |\omega_{B,nh}^\alpha|^2 dV_t + \varepsilon \int_S |\nabla^\varphi \omega_{B,nh}^\alpha|^2 dV_t + \int_S (B \cdot \nabla^\varphi) \omega_{B,nh}^\alpha \cdot \omega_{v,nh}^\alpha dV_t = \int_S E \cdot \omega_{B,nh}^\alpha dV_t$$

I) $\|Z^\alpha(\omega_v \cdot \nabla^\varphi B)\|, \|Z^\alpha(\omega_B \cdot \nabla^\varphi v)\|$ estimate.

Simply we get

$$\begin{aligned} \|Z^\alpha(\omega_v \cdot \nabla^\varphi B)\| &\leq \Lambda_{\infty,m} \left(\|\omega_v\|_{m-1} + \|B\|_m + |h|_{m-\frac{1}{2}} \right) \\ \|Z^\alpha(\omega_B \cdot \nabla^\varphi v)\| &\leq \Lambda_{\infty,m} \left(\|\omega_B\|_{m-1} + \|v\|_m + |h|_{m-\frac{1}{2}} \right) \end{aligned}$$

II) $\left| \int_S \bar{\mathcal{C}}_S^2 \cdot \omega_{B,nh}^\alpha dV_t \right|$ estimate.

As like in $S_n^{v,B}$ estimate,

$$\left| \int_S \bar{\mathcal{C}}_S^2 \cdot \omega_{B,nh}^\alpha dV_t \right| \leq \Lambda_0 \left(\sqrt{\varepsilon} \|\nabla^\varphi \omega_{B,nh}^\alpha\| + \|\omega_B\|_{m-1} \right) \left(\sqrt{\varepsilon} \|\nabla \omega_B\|_{m-2} + \|\omega_B\|_{m-1} + \Lambda_{\infty,m} \left(|h|_{m-\frac{1}{2}} + \sqrt{\varepsilon} |h|_{m+\frac{1}{2}} \right) \right)$$

III) $\|\bar{\mathcal{C}}_{S_y}^1\|, \|\bar{\mathcal{C}}_{S_y}^3\|$ estimate.

This is also similar to previous $S_n^{v,B}$ estimate,

$$\begin{aligned} \|\bar{\mathcal{C}}_{S_y}^1\| &\leq \Lambda_{\infty,m} (\|\omega_B\|_{m-1} + \|v\|_m + |h|_m) \\ \|\bar{\mathcal{C}}_{S_y}^3\| &\leq \Lambda_{\infty,m} (\|\omega_v\|_{m-1} + \|B\|_m + |h|_m) \end{aligned}$$

IV) $\|\bar{\mathcal{C}}_{S_z}^1\|, \|\bar{\mathcal{C}}_{S_z}^3\|$ estimate.

We skip the detail, since it is nearly same as previous proposition.

$$(10.29) \quad \|\mathcal{C}_{S_z}^1\| \leq \Lambda_{\infty,m} (\|\omega_v\|_{m-1} + \|B\|_{E^m} + |h|_m)$$

$$(10.30) \quad \|\mathcal{C}_{S_z}^3\| \leq \Lambda_{\infty,m} (\|\omega_B\|_{m-1} + \|v\|_{E^m} + |h|_m)$$

Hence as like previous proposition, we get the result. □

Summing above two propositions 10.1 and 10.2, we get

Proposition 10.3. *Non-homogeneous part estimate.*

$$\begin{aligned} (10.31) \quad &\|\omega_{v,nh}^{m-1}(t)\|^2 + \|\omega_{B,nh}^{m-1}(t)\|^2 + \varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{v,nh}^{m-1}|^2 dV_t ds + \varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{B,nh}^{m-1}|^2 dV_t ds \\ &\leq \Lambda_0 \left(\|\omega_{v,nh}^{m-1}(0)\|^2 + \|\omega_{B,nh}^{m-1}(0)\|^2 \right) + \varepsilon \int_0^t \Lambda_{\infty,m} (\|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) ds \\ &\quad + \int_0^t \Lambda_{\infty,m} \left(\|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\omega_v\|_{m-1}^2 + \|\omega_B\|_{m-1}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) ds \end{aligned}$$

Hence, ω_{nh} has zero boundary condition, we get the L^∞ type energy estimate. Main difficulty of this section is how to estimate ω_h which has nonzero boundary condition. In this case, we get only L_t^4 type estimate.

10.2. Homogeneous estimate-1. We estimate $\omega_{v,h}^{\alpha,2}, \omega_{B,h}^{\alpha,2}$. From (10.14) and (10.15), we can easily know that we get similar estimate to proposition 10.3.

Proposition 10.4. *Non-homogeneous part estimate.*

$$(10.32) \quad \begin{aligned} & \|\omega_{v,nh}^{m-1,2}(t)\|^2 + \|\omega_{B,nh}^{m-1,2}(t)\|^2 + \varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{v,nh}^{m-1,2}|^2 dV_t ds + \varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{B,nh}^{m-1,2}|^2 dV_t ds \\ & \leq \Lambda_0 \left(\|\omega_{v,nh}^{m-1,2}(0)\|^2 + \|\omega_{B,nh}^{m-1,2}(0)\|^2 \right) + \varepsilon \int_0^t \Lambda_{\infty,m} (\|\nabla S_n^v\|_{m-2}^2 + \|\nabla S_n^B\|_{m-2}^2) ds \\ & \quad + \int_0^t \Lambda_{\infty,m} \left(\|v\|_{E^m}^2 + \|B\|_{E^m}^2 + \|\omega_v\|_{m-1}^2 + \|\omega_B\|_{m-1}^2 + |h|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) ds \end{aligned}$$

10.3. Homogeneous estimate-2. In this section, we estimate $\omega_{v,h}^{\alpha,1}, \omega_{B,h}^{\alpha,1}$ those have nonzero boundary condition as we see in (10.12) and (10.13). For motivation for this estimate see [1]. Now, suppose that

$$z = f(t, x, y), \quad z = \bar{f}(t, x, y)$$

is a boundary function of X map and Y map from Ω_0 . (So, $f(0, x, y) = \bar{f}(0, x, y)$)

$$\begin{aligned} f(t, x, y) &= f(0, x, y) + \int_0^t (\partial_t f)(s, x, y) ds = \bar{f}(0, x, y) + \int_0^t \hat{e}_3 \cdot u^b(s, x, y) ds \\ &= \bar{f}(0, x, y) + \int_0^t \hat{e}_3 \cdot (u - H)^b(s, x, y) ds = \bar{f}(0, x, y) + \int_0^t (\partial_t \bar{f})(s, x, y) ds = \bar{f}(t, x, y) \end{aligned}$$

So, we can construct new maps Y_i ($i = 1, 2$)

$$(10.33) \quad \begin{aligned} Y_1 : \Omega_0 &\rightarrow \Omega_t, \quad \partial_t Y_1(t, x) = (u - H)(t, Y_1(t, x)), \quad Y_1(0, x) = x \\ Y_2 : \Omega_0 &\rightarrow \Omega_t, \quad \partial_t Y_2(t, x) = (u + H)(t, Y_2(t, x)), \quad Y_2(0, x) = x \end{aligned}$$

Now, we use equation for $\omega_{v,h}^\alpha, \omega_{B,h}^\alpha$, to get

$$(10.34) \quad \partial_t^\varphi (\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) + (v - B) \cdot \nabla^\varphi (\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) - \varepsilon \Delta^\varphi (\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) = 0$$

with initial and boundary condition,

$$(\omega_{v,h}^\alpha + \omega_{B,h}^\alpha)^b = (Z^\alpha \omega_v)^b + (Z^\alpha \omega_B)^b, \quad (\omega_{v,h}^\alpha + \omega_{B,h}^\alpha)_{t=0} = 0$$

and

$$(10.35) \quad \partial_t^\varphi (\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) + (v + B) \cdot \nabla^\varphi (\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) - \varepsilon \Delta^\varphi (\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) = 0$$

with initial and boundary condition,

$$(\omega_{v,h}^\alpha - \omega_{B,h}^\alpha)^b = (Z^\alpha \omega_v)^b - (Z^\alpha \omega_B)^b, \quad (\omega_{v,h}^\alpha - \omega_{B,h}^\alpha)_{t=0} = 0$$

We transform these equations into

$$(10.36) \quad \begin{aligned} & \partial_t ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1}) + ((v - B) \circ \Phi^{-1}) \cdot \nabla ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1}) - \varepsilon \Delta ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1}) = 0 \\ & \partial_t ((\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) \circ \Phi^{-1}) + (u - H) \cdot \nabla ((\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) \circ \Phi^{-1}) - \varepsilon \Delta ((\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) \circ \Phi^{-1}) = 0 \\ & \partial_t ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1} \circ X) - \varepsilon \frac{1}{a_0} \partial_i (a_{ij} \partial_j ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1} \circ X)) = 0 \end{aligned}$$

Write

$$\begin{aligned} & ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1} \circ X) \doteq \mathcal{W}_+ \\ & \partial_t \mathcal{W}_+ - \varepsilon \frac{1}{a_0} \partial_i (a_{ij} \partial_j \mathcal{W}_+) = 0 \end{aligned}$$

Again, and similarly for minus case, (a,b are defined for Y_1, Y_2 respectively)

$$\Omega_+^\alpha \doteq e^{-\gamma t} \mathcal{W}_+, \quad \Omega_-^\alpha \doteq e^{-\gamma t} \mathcal{W}_-$$

We get

$$(10.37) \quad a_0(\partial_t \Omega_+^\alpha + \gamma \Omega_+^\alpha) - \varepsilon \partial_i (a_{ij} \partial_j \Omega_+^\alpha) = 0$$

with boundary condition

$$\Omega_+^\alpha|_{z=h_0} = e^{-\gamma t} ((\omega_{v,h}^\alpha + \omega_{B,h}^\alpha) \circ \Phi^{-1} \circ X)(t, y, h_0(y))$$

and similarly,

$$(10.38) \quad b_0(\partial_t \Omega_-^\alpha + \gamma \Omega_-^\alpha) - \varepsilon \partial_i (b_{ij} \partial_j \Omega_-^\alpha) = 0$$

with boundary condition

$$\Omega_-^\alpha|_{z=h_0} = e^{-\gamma t} ((\omega_{v,h}^\alpha - \omega_{B,h}^\alpha) \circ \Phi^{-1} \circ X)(t, y, h_0(y))$$

We will just use theorem 10.6 in [1]. To do this we should first show that Y_1, Y_2 satisfy Lemma 10.5 in [1].

Lemma 10.5. *Let's assume that for $T \in [0, T^\varepsilon]$, $T^\varepsilon \leq 1$, there exist $M > 0$ such that the following holds.*

$$\sup_{[0, T]} \Lambda_{\infty, 6}(t) + \int_0^T (\varepsilon \|\nabla \mathcal{V}^6\|^2 + \varepsilon \|\nabla \mathcal{B}^6\|^2 + \varepsilon \|\nabla S_n^v\|^2 + \varepsilon \|\nabla S_n^B\|^2) \leq M$$

Under above assumption, for $t \in [0, T]$ and $i = 1, 2$ we have the following estimates.

$$(10.39) \quad |J_i(t, x)|_{W^{1, \infty}} + |1/J_i(t, x)|_{W^{1, \infty}} \leq \Lambda_0$$

$$(10.40) \quad |\nabla Y_i(t)|_{L^\infty} + |\partial_t \nabla Y_i(t)|_{L^\infty} \leq \Lambda_0 e^{\Lambda(M)t}$$

$$(10.41) \quad |\nabla Y_i(t)|_{W^{1, \infty}} + |\partial_t \nabla Y_i(t)|_{W^{1, \infty}} \leq \Lambda(M) e^{\Lambda(M)t}$$

$$(10.42) \quad \sqrt{\varepsilon} \|\nabla^2 Y_i\|_{L^\infty} + \sqrt{\varepsilon} \|\partial_t \nabla^2 Y_i\|_{L^\infty} \leq \Lambda(M) (1+t)^2 e^{\Lambda(M)t}$$

Proof. 1) First one comes from the fact that $J_i(t, x) = J_i(0, x)$, since u, H are both divergence free.

2) Secondly,

$$\partial_t DY_i = D(v \mp B) D\Phi^{-1} DY_i$$

Taking L^∞ and using Gronwall's inequality gives

$$|\nabla Y_i(t, \cdot)|_{L^\infty} \leq \Lambda_0 e^{\Lambda(M)t}$$

Again taking L^∞ to above chain rule, and using the result, we get

$$|\partial_t \nabla Y_i(t, \cdot)|_{L^\infty} \leq \Lambda_0 e^{\Lambda(M)t}$$

These two inequality gives second one.

3) We take conormal derivative Z to above chain rule,

$$\partial_t Z(DY_i) = ZD(v \mp B) D\Phi^{-1} DY_i + D(v \mp B) ZD\Phi^{-1} DY_i + D(v \mp B) D\Phi^{-1} ZDY_i$$

Now we use above results, and Gronwall's inequality to get

$$|\nabla Y_i(t, \cdot)|_{W^{1, \infty}} \leq \Lambda(M) e^{\Lambda(M)t}$$

Again using conormally differentiated chain rule,

$$|\partial_t \nabla Y_i(t, \cdot)|_{W^{1, \infty}} \leq \Lambda(M) e^{\Lambda(M)t}$$

4) We take $\sqrt{\varepsilon} \nabla$ to chain rule, to get

$$\sqrt{\varepsilon} \partial_t D^2 Y_i = \sqrt{\varepsilon} D^2(v \mp B) D\Phi^{-1} DY_i + \sqrt{\varepsilon} D(v \mp B) D^2 \Phi^{-1} DY_i + \sqrt{\varepsilon} D(v \mp B) D\Phi^{-1} D^2 Y_i$$

We use proposition 9.9 and Gronwall's inequality, then

$$\sqrt{\varepsilon} \|\nabla^2 Y_i\|_{L^\infty} \leq \Lambda(M) (1+t)^2 e^{\Lambda(M)t} + \Lambda(M) \int_0^t \sqrt{\varepsilon} \|\nabla^2 Y_i(s)\|_{L^\infty}$$

$$\sqrt{\varepsilon} \|\nabla^2 Y_i\|_{L^\infty} \leq \Lambda(M) (1+t)^2 e^{\Lambda(M)t}$$

Using this result, chain rule, and Gronwall, t-derivatived is also.

$$\sqrt{\varepsilon} \|\partial_t \nabla^2 Y_i\|_{L^\infty} \leq \Lambda(M) (1+t)^2 e^{\Lambda(M)t}$$

□

Since both Y_i has same control as like in [1], we can use Theorem 10.6 in [1], to get,

Theorem 10.6. *These exist γ_0 such that for $\gamma \geq \gamma_0$ solution Ω_{\pm}^{α} satisfy the following estimate.*

$$(10.43) \quad \|\Omega_{\pm}^{m-1}\|_{H^{\frac{1}{4}}(0,T;L^2)}^2 \leq \Lambda(M)\sqrt{\varepsilon} \int_0^T |(\Omega_{\pm}^{m-1})^b|_{L^2}^2$$

Proof. See [1]. □

Now we state estimate for homogeneous part.

Proposition 10.7. *Under above assumption, we get, for $|\alpha| \leq m-1$,*

$$(10.44) \quad \|\omega_{v,h}^{\alpha}\|_{L^4(0,T;L^2)}^2, \|\omega_{B,h}^{\alpha}\|_{L^4(0,T;L^2)}^2 \leq \Lambda(M) \int_0^T (\|\mathcal{V}^m\|^2 + \|\mathcal{B}^m\|^2 + |h|_m^2) + \frac{\varepsilon}{2} \int_0^T (\|\nabla \mathcal{V}^m\|^2 + \|\nabla \mathcal{B}^m\|^2)$$

Proof. By 1-dimensional sobolev embedding, (for first inequality)

$$\|\Omega_{\pm}^{\alpha}\|_{L^4(0,T;L^2)}^2 \leq C \|\Omega_{\pm}^{m-1}\|_{H^{\frac{1}{4}}(0,T;L^2)}^2 \leq \Lambda(M)\sqrt{\varepsilon} \int_0^T e^{-2\gamma t} ((\omega_{v,h}^{\alpha} \pm \omega_{B,h}^{\alpha}) \circ \Phi^{-1} \circ X)^2$$

For general sobolev embedding, C may blow up as $T \rightarrow 0$. But here, C is independent to T . This is because we use sobolev embedding on \mathbb{R}_+ . (We first extend $[0, T]$ to half line. Then take inf function for all such functions. See [1]) Next, we can change variable to function on S using Φ and estimate J_i of above lemma, to get

$$\begin{aligned} \|\omega_{v,h}^{\alpha} \pm \omega_{B,h}^{\alpha}\|_{L^4(0,T;L^2)}^2 &\leq \Lambda(M)\sqrt{\varepsilon} \int_0^T \|(Z^{\alpha}\omega_v)^b \pm (Z^{\alpha}\omega_B)^b\|_{m-1}^2 \\ &\leq \Lambda(M)\sqrt{\varepsilon} \int_0^T (\|(Z^{\alpha}\omega_v)^b\|^2 + \|(Z^{\alpha}\omega_B)^b\|^2) \end{aligned}$$

By (10.7), we finish the proof. □

11. UNIFORM EXISTENCE

Now we prove first uniform regularity theorem. For sufficiently big $m \geq 6$, let initial data

$$(11.1) \quad \begin{aligned} \mathcal{I}_m(0) &\doteq \|v_0\|_{E^m} + \|B_0\|_{E^m} + |h_0|_m + \sqrt{\varepsilon}|h_0|_{m+\frac{1}{2}} + \|v_0\|_{E^{2,\infty}} + \|B_0\|_{E^{2,\infty}} \\ &\quad + \sqrt{\varepsilon}\|\partial_{zz}v_0\|_{L^\infty} + \sqrt{\varepsilon}\|\partial_{zz}B_0\|_{L^\infty} \leq \infty \end{aligned}$$

We can regularize this data so that $v_0^\delta, B_0^\delta \in W_2^{l+1} \times W_2^{l+1}$ then we can use the result of [22] to guarantee the existence of the solution in $H^{l+2,l/2+1} \times H^{l+2,l/2+1}$, $l \in (\frac{1}{2}, 1)$, where space $H^{l+2,l/2+1}$ is defined as

$$\|u\|_{H^{l+2,l/2+1}(Q_T)}^2 = \|u\|_{Q_T}^{(l+2)^2} + \sup_{t < T} \|u(t)\|_{W_2^{l+1}(\Omega)}^2$$

On the right hand side, $\|u\|_{Q_T}^{(l+2)^2}$ is in fact,

$$\|u\|_{Q_T}^{(l+2)^2} = \|u\|_{Q_T}^{(l)^2} + \sum_{|s|=2} \|D_x^s u\|_{Q_T}^{(l)^2} + \sum_{|s|=0}^1 \|D_x^s u\|_{L^2(Q_T)}^2$$

where

$$\|u\|_{Q_T}^{(l)^2} = \|u\|_{W_2^{l,l/2}(Q_T)}^2 + T^{-l} \|u\|_{L^2(Q_T)}^2$$

At last, we are suffice to identify anisotropic Sobolev-Slobodetskii space $W_2^{l,l/2}(Q_T)$ which is defined as follow.(see [22] or [7])

Definition 11.1. By $W_2^l(\Omega)$, we mean

$$\|u\|_{W_2^l(\Omega)}^2 = \sum_{0 \leq |\alpha| < l} \|D^\alpha u\|_{L^2(\Omega)}^2 + \|u\|_{W_2^l(\Omega)}^2$$

where

$$\|u\|_{W_2^l(\Omega)}^2 = \begin{cases} \sum_{|\alpha|=l} \|D^\alpha u\|_{L^2(\Omega)}^2 & \text{if } l \in \mathbb{Z}, \\ \sum_{|\alpha|=l} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2l}} & \text{if } l = [l] + l \notin \mathbb{Z}, 0 < l < 1 \end{cases}$$

We also define the anisotropic space $W_2^{l,l/2}(Q_T = \Omega \times (0, T))$ as

$$\begin{aligned} \|u\|_{W_2^{l,l/2}(Q_T)}^2 &= \|u\|_{W_2^{l,0}(Q_T)}^2 + \|u\|_{W_2^{0,l/2}(Q_T)}^2 \\ &= \int_0^T \|u(t, \cdot)\|_{W_2^l(\Omega)}^2 dt + \int_\Omega \|u(\cdot, x)\|_{W_2^{l/2}(0,T)}^2 dx \end{aligned}$$

Next we define, $H_\gamma^{l,l/2}(Q_T)$, $\gamma \geq 0$, by

$$\|u\|_{H_\gamma^{l,l/2}(Q_T)}^2 = \|u\|_{H_\gamma^{l,0}(Q_T)}^2 + \|u\|_{H_\gamma^{0,l/2}(Q_T)}^2$$

where

$$\begin{aligned} \|u\|_{H_\gamma^{l,0}(Q_T)}^2 &= \int_0^T e^{-2\gamma t} \|u(t, \cdot)\|_{W_2^l(\Omega)}^2 dt \\ \|u\|_{H_\gamma^{0,l/2}(Q_T)}^2 &= \gamma^l \int_0^T e^{-2\gamma t} \|u(t, \cdot)\|_{L^2(\Omega)}^2 dt \\ &+ \int_0^T e^{-2\gamma t} dt \int_0^\infty \left\| \left(\frac{\partial}{\partial t} \right)^k u_0(t, \cdot) - \left(\frac{\partial}{\partial t} \right)^k u_0(t - \tau, \cdot) \right\|_{L^2(\Omega)}^2 \frac{d\tau}{\tau^{1+l-2k}} \end{aligned}$$

if $l/2$ is not an integer, $k = [l/2]$, $u_0(t, x) = u(t, x)(t > 0)$, $u_0(t, x) = 0(t < 0)$. If $l/2$ is an integer, then the double integral in the norm should be replaced by

$$\int_{-\infty}^T e^{-2\gamma t} \left\| \left(\frac{\partial}{\partial t} \right)^{l/2} u(t, \cdot) \right\|_{L^2(\Omega)}^2 dt$$

and $\left(\frac{\partial}{\partial t} \right)^j u|_{t=0}$ ($j = 0, 1, 2, \dots, l/2 - 1$) should be satisfied. We also introduce the space $H_\gamma^{l+1/2,1/2,l/2}(\Gamma_T)$ by

$$\begin{aligned} &\|u\|_{H_\gamma^{l+1/2,1/2,l/2}(\Gamma_T)}^2 \\ &= \int_0^T e^{-2\gamma t} \left(\|u\|_{W_2^{l+1/2}(\Gamma)}^2 + \gamma^l \|u\|_{W_2^{1/2}(\Gamma)}^2 \right) dt \\ &+ \int_0^T e^{-2\gamma t} dt \int_0^\infty \left\| \left(\frac{\partial}{\partial t} \right)^k u_0(t, \cdot) - \left(\frac{\partial}{\partial t} \right)^k u_0(t - \tau, \cdot) \right\|_{W_2^{1/2}(\Gamma)}^2 \frac{d\tau}{\tau^{1+l-2k}} \end{aligned}$$

if $l/2$ is not an integer, $k = [l/2]$, $u_0(t, x) = u(t, x)(t > 0)$, $u_0(t, x) = 0(t < 0)$. If $l/2$ is an integer, then the double integral in the norm should be replaced by

$$\int_{-\infty}^T e^{-2\gamma t} \left\| \left(\frac{\partial}{\partial t} \right)^{l/2} u(t, \cdot) \right\|_{W_2^{1/2}(\Gamma)}^2 dt$$

and $\left(\frac{\partial}{\partial t} \right)^j u|_{t=0}$ ($j = 0, 1, 2, \dots, l/2 - 1$) should be satisfied.

Now, from the above initial condition, we get local time $T^{\varepsilon, \delta}$ such that $\forall T \leq T^{\varepsilon, \delta}$,

(11.2)

$$\begin{aligned} \mathcal{N}_m(T) &= \sup_{[0,T]} \left(\|v(t)\|_m^2 + \|B(t)\|_m^2 + |h(t)|_m^2 + \varepsilon |h(t)|_{m+\frac{1}{2}}^2 + \|v(t)\|_{E^{2,\infty}}^2 + \|B(t)\|_{E^{2,\infty}}^2 + \varepsilon \|\partial_{zz}v(t)\|_{L^\infty}^2 + \varepsilon \|\partial_{zz}B(t)\|_{L^\infty}^2 \right) \\ &+ \left(\|\partial_z v\|_{L^4([0,T];H_{co}^{m-1})}^2 + \|\partial_z B\|_{L^4([0,T];H_{co}^{m-1})}^2 \right) + \varepsilon \int_0^T \|\nabla v\|_m^2 + \varepsilon \int_0^T \|\nabla B\|_m^2 + \varepsilon \int_0^T \|\nabla \partial_z v\|_{m-2}^2 + \varepsilon \int_0^T \|\nabla \partial_z B\|_{m-2}^2 \leq \infty \end{aligned}$$

See more remark in [1] middle of page 74 later.

It is equivalent to control (instead of $\mathcal{N}_m(T)$),

(11.3)

$$\begin{aligned} \mathcal{E}_m(T) &= \sup_{[0,T]} \left(\|\mathcal{V}^m(t)\|^2 + \|\mathcal{B}^m(t)\|^2 + |h(t)|_m^2 + \varepsilon |h(t)|_{m+\frac{1}{2}}^2 + \|S_n^v(t)\|_{E^{2,\infty}}^2 + \|S_n^B(t)\|_{E^{2,\infty}}^2 + \varepsilon \|\partial_z S_n^v(t)\|_{L^\infty}^2 + \varepsilon \|\partial_z S_n^B(t)\|_{L^\infty}^2 \right) \\ &+ \left(\|\omega_v\|_{L^4([0,T];H_{co}^{m-1})}^2 + \|\omega_B\|_{L^4([0,T];H_{co}^{m-1})}^2 \right) + \varepsilon \int_0^T \|\nabla \mathcal{V}^m\|^2 + \varepsilon \int_0^T \|\nabla \mathcal{B}^m\|^2 + \varepsilon \int_0^T \|\nabla S_n^v\|_{m-2}^2 + \varepsilon \int_0^T \|\nabla S_n^B\|_{m-2}^2 \leq \infty \end{aligned}$$

Two parameter R, c_0 , so that $\frac{1}{c_0} \ll R$, define $T_*^{\varepsilon, \delta}$, define,

$$(11.4) \quad T_*^{\varepsilon, \delta} = \sup \left\{ T \in [0, 1] \mid \mathcal{E}_m(t) \leq R, \quad |h(t)|_{2, \infty} \leq \frac{1}{c_0}, \quad \partial_z \varphi(t) \geq c_0, \quad g - (\partial_z^\varphi q^E)|_{z=0} \geq \frac{c_0}{2}, \quad \forall t \in [0, T] \right\}$$

Now we combine our propositions and corollaries to get uniform energy estimate for both ε and δ . Note that from corollary 9.1, for $T \leq T_*^{\varepsilon, \delta}$,

$$\Lambda_{m, \infty}(T) \leq \Lambda(R)$$

Using (7.15), (7.16), (8.11), (9.47), (9.50), we obtain,

$$\mathcal{E}_m(T) \leq \Lambda\left(\frac{1}{c_0}, \mathcal{I}_m(0)\right) + \Lambda(R)\sqrt{T} + \Lambda(R) \int_0^T |(\partial_z \partial_t q^E)^b|_{L^\infty}$$

Using proposition 5.6 again,

$$(11.5) \quad E_m(T) \leq \Lambda\left(\frac{1}{c_0}, \mathcal{I}_m(0)\right) + \Lambda(R)\sqrt{T}$$

which is (ε, δ) -independent inequality. Moreover for about conditions in definition of $T_*^{\varepsilon, \delta}$, for $T \leq T_*^{\varepsilon, \delta}$,

$$(11.6) \quad |h(t)|_{2, \infty} \leq |h(0)|_{2, \infty} + \Lambda(R)T$$

$$(11.7) \quad \partial_z \varphi(t) \geq 1 - \int_0^t \|\partial_t \nabla \eta\|_{L^\infty} \geq 1 - \Lambda(R)T$$

$$(11.8) \quad g - (\partial_z^\varphi q^E)^b(t) \geq g - (\partial_z^\varphi q^E)|_{z=0} - \Lambda(R) \int_0^t (1 + |(\partial_t \partial_z q^E)^b|_{L^\infty}) \leq g - (\partial_z^\varphi q^E)|_{z=0} - \Lambda(R)\sqrt{t}$$

Note that above four inequalities are all (ε, δ) -independent inequality. Hence we can choose proper $R = \Lambda(\mathcal{I}_m(0), |h|_{2, \infty})$ so that we can pick T_* which is (ε, δ) -independent and for $T \leq \min(T_*, T_*^{\varepsilon, \delta})$, four inequalities in (11.4) are satisfied. Now we can send regularizing parameter $\delta \rightarrow 0$, to get uniform time interval T_* for initial data $\mathcal{I}_\#(t)$. (For each δ , $\mathcal{N}_m(T_*)$ is uniformly bounded in δ so we can use strong compactness.)

12. UNIQUENESS

12.1. Uniqueness of Theorem 1.4. In previous section, we proved existence of viscous system (1.4). In this subsection we prove uniqueness of the system. We suppose two solutions $(v^1, B^1, \varphi^1, q^1), (v^2, B^2, \varphi^2, q^2)$ with same initial condition. We will perform L^2 energy estimate (zero order estimate) and use Gronwall's inequality to show that L^2 norm of differences are locally zero. As usual, we write

$$\bar{v}^\varepsilon = v_1^\varepsilon - v_2^\varepsilon, \quad \bar{B}^\varepsilon = B_1^\varepsilon - B_2^\varepsilon, \quad \bar{h}^\varepsilon = h_1^\varepsilon - h_2^\varepsilon, \quad \bar{q}^\varepsilon = q_1^\varepsilon - q_2^\varepsilon$$

with initial condition $\bar{v}(0) = \bar{B}(0) = 0, \bar{h}(0) = 0$. Let both solutions satisfy on $[0, T^\varepsilon]$,

$$\mathcal{N}_m^i(T^\varepsilon) \leq R, \quad i = 1, 2$$

By divergence free condition, $\nabla^{\varphi^i} \cdot v_i^\varepsilon = 0, \quad i = 1, 2$,

$$(\partial_t + v_{y,i}^\varepsilon \cdot \nabla_y + V_{z,i}^\varepsilon \partial_z) v_i^\varepsilon + \nabla^{\varphi^i} q_i^\varepsilon - \varepsilon \Delta^{\varphi^i} v_i^\varepsilon = (B_i^\varepsilon \cdot \nabla^{\varphi^i}) B_i$$

Then we have equation about $(\bar{v}^\varepsilon, \bar{B}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$. First for Navier-Stokes,

$$(12.1) \quad (\partial_t + v_{y,1}^\varepsilon \cdot \nabla_y + V_{z,1}^\varepsilon \partial_z) \bar{v}^\varepsilon + \nabla^{\varphi^1} \bar{q}^\varepsilon - \varepsilon \Delta^{\varphi^1} \bar{v}^\varepsilon - (B_1^\varepsilon \cdot P_1^* \nabla) \bar{B}^\varepsilon = F$$

where

$$\begin{aligned} F = & (v_{y,2}^\varepsilon - v_{y,1}^\varepsilon) \cdot \nabla_y v_2^\varepsilon + (V_{z,2}^\varepsilon - V_{z,1}^\varepsilon) \partial_z v_2^\varepsilon - \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) (P_1^* \nabla q_2^\varepsilon) + \frac{1}{\partial_z \varphi_2^\varepsilon} ((P_2 - P_1)^* \nabla q_2^\varepsilon) \\ & + \varepsilon \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) \nabla \cdot (E_1 \nabla v_2^\varepsilon) + \varepsilon \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot ((E_2 - E_1) \nabla v_2^\varepsilon) + (\bar{B}^\varepsilon \cdot P_2^* \nabla) B_2^\varepsilon + (B_1^\varepsilon \cdot (P_1^* - P_2^*) \nabla) B_2^\varepsilon \end{aligned}$$

For Faraday's law, similarly as above, we have

$$(12.2) \quad (\partial_t + v_{y,1}^\varepsilon \cdot \nabla_y + V_{z,1}^\varepsilon \partial_z) \bar{B}^\varepsilon - \varepsilon \Delta^{\varphi^1} \bar{B}^\varepsilon - (B_1^\varepsilon \cdot P_1^* \nabla) \bar{v}^\varepsilon = E$$

where

$$E = (v_{y,2}^\varepsilon - v_{y,1}^\varepsilon) \cdot \nabla_y B_2^\varepsilon + (V_{z,2}^\varepsilon - V_{z,1}^\varepsilon) \partial_z B_2^\varepsilon + \varepsilon \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) \nabla \cdot (E_1 \nabla B_2^\varepsilon) + \varepsilon \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot ((E_2 - E_1) \nabla B_2^\varepsilon) \\ + (\bar{B}^\varepsilon \cdot P_2^* \nabla) v_2^\varepsilon + (B_1^\varepsilon \cdot (P_1^* - P_2^*) \nabla) v_2^\varepsilon$$

For divergence-free condition,

$$(12.3) \quad \nabla^{\varphi_1} \cdot \bar{v}^\varepsilon = - \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) \nabla \cdot (P_1 v_2^\varepsilon) - \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot ((P_2 - P_1) v_2^\varepsilon), \text{ same for } B$$

For Kinematic boundary condition,

$$(12.4) \quad \partial_t \bar{h}^\varepsilon - (v^\varepsilon)_{y,1}^b \cdot \nabla h + ((v_{z,1}^\varepsilon)^b - (v_{z,2}^\varepsilon)^b) = - ((v_{y,2}^\varepsilon)^b - (v_{y,1}^\varepsilon)^b) \cdot \nabla h_2^\varepsilon$$

Continuity of stress tensor condition becomes,

$$(12.5) \quad \bar{q}^\varepsilon \hat{n}_1 - 2\varepsilon (S^{\varepsilon_1} \bar{v}^\varepsilon) \hat{n}_1 - g \bar{h}^\varepsilon = 2\varepsilon ((S^{\varphi_1} - S^{\varphi_2}) v_2^\varepsilon) \hat{n}_1 + 2\varepsilon (S^{\varphi_2} v_2^\varepsilon) (\hat{n}_1 - \hat{n}_2)$$

Using (12.1)-(12.5), we get L^2 - energy estimate. Details are nearly same as high order estimate which was shown throughout this paper. (Since initial condition is zero, no initial term appear on right hand side)

$$(12.6) \quad \|\bar{v}^\varepsilon(t)\|_{L^2}^2 + \|\bar{B}^\varepsilon(t)\|_{L^2}^2 + \|\bar{h}^\varepsilon(t)\|_{L^2}^2 + \varepsilon \int_0^t \left(\|\nabla \bar{v}^\varepsilon\|_{L^2}^2 + \|\nabla \bar{B}^\varepsilon\|_{L^2}^2 \right) ds \\ \leq \Lambda(R) \int_0^t \left(\|\bar{v}^\varepsilon(s)\|_{L^2}^2 + \|\bar{B}^\varepsilon(s)\|_{L^2}^2 + \|\bar{h}^\varepsilon(s)\|_{H^{\frac{1}{2}}}^2 + \|\nabla \bar{q}^\varepsilon\|_{L(S)} \|\bar{v}^\varepsilon\|_{L(S)} \right) ds$$

Using pressure estimate, we also get,

$$\|\nabla \bar{q}^\varepsilon\|_{L(S)} \leq \Lambda(R) (\|\bar{h}^\varepsilon\|_{H^{1/2}} + \|\bar{v}^\varepsilon\|_{H^1(S)} + \|\bar{B}^\varepsilon\|_{H^1(S)})$$

Now with help of Lemma 3.9, we get the result.4

$$(12.7) \quad \|\bar{v}^\varepsilon(t)\|_{L^2}^2 + \|\bar{B}^\varepsilon(t)\|_{L^2}^2 + \|\bar{h}^\varepsilon(t)\|_{L^2}^2 + \sqrt{\varepsilon} \|\bar{h}^\varepsilon(t)\|_{H^{1/2}}^2 + \varepsilon \int_0^t \left(\|\nabla \bar{v}^\varepsilon\|_{L^2}^2 + \|\nabla \bar{B}^\varepsilon\|_{L^2}^2 \right) ds \\ \leq \Lambda(R) \int_0^t \left(\|\bar{v}^\varepsilon(s)\|_{L^2}^2 + \|\bar{B}^\varepsilon(s)\|_{L^2}^2 + \|\bar{h}^\varepsilon(s)\|_{H^{\frac{1}{2}}}^2 \right) ds$$

In above equations for $(\bar{v}^\varepsilon, \bar{B}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$, right hand side does not have low order than L^2 energy. However we have uniform bound of m-order energy, so we can collect high order terms into $\Lambda(R)$. Also note that this is uniqueness for every fixed ε . This means this estimate is not uniform in ε .

12.2. Uniqueness of Theorem 1.5. For two solutions $(v_1, B_1, h_1, q_1), (v_2, B_2, h_2, q_2)$ with same initial condition. Suppose,

$$(12.8) \quad \sup_{[0,T]} \left(\|v_i\|_m + \|\partial_z v_i\|_{m-1} + \|\partial_z v_i\|_{1,\infty} + \|B_i\|_m + \|\partial_z B_i\|_{m-1} + \|\partial_z B_i\|_{1,\infty} + \|h_i\|_m \right) \leq R, \quad i = 1, 2$$

Define $\bar{v} \doteq v_1 - v_2$, $\bar{B} \doteq B_1 - B_2$, $\bar{h} \doteq h_1 - h_2$, $\bar{q} \doteq q_1 - q_2$ and we write equation of $(\bar{v}, \bar{h}, \bar{q})$, as before. Euler equation becomes,

$$(12.9) \quad (\partial_t + v_{y,1} \cdot \nabla_y + V_{z,1} \partial_z) \bar{v} + \nabla^{\varphi_1} \bar{q} - (B_1 \cdot \nabla^{\varphi_1}) \bar{B} = F'$$

where

$$F' = (v_{y,2} - v_{y,1}) \cdot \nabla_y v_2 + (V_{z,2} - V_{z,1}) \partial_z v_2 - \left(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1} \right) (P_1^* \nabla q_2) + \frac{1}{\partial_z \varphi_2} ((P_2 - P_1)^* \nabla q_2) \\ + (\bar{B} \cdot P_2^* \nabla) B_2 + (B_1 \cdot (P_1^* - P_2^*) \nabla) B_2$$

For Faraday's law, similarly,

$$(12.10) \quad (\partial_t + v_{y,1} \cdot \nabla_y + V_{z,1} \partial_z) \bar{v} - (B_1 \cdot \nabla^{\varphi_1}) \bar{v} = E'$$

where

$$E' = (v_{y,2} - v_{y,1}) \cdot \nabla_y v_2 + (V_{z,2} - V_{z,1}) \partial_z v_2 + (\bar{B} \cdot P_2^* \nabla) v_2 + (B_1 \cdot (P_1^* - P_2^*) \nabla) v_2$$

For divergence-free condition, (same for B)

$$(12.11) \quad \nabla^{\varphi_1} \cdot \bar{v} = - \left(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1} \right) \nabla \cdot (P_1 v_2) - \frac{1}{\partial_z \varphi_2} \nabla \cdot ((P_2 - P_1) v_2)$$

For Kinematic boundary condition,

$$(12.12) \quad \partial_t \bar{h} - v_{y,1}^b \cdot \nabla h + (v_{z,1}^b - v_{z,2}^b) = - (v_{y,2}^b - v_{y,1}^b) \cdot \nabla h_2$$

Continuity of stress tensor condition becomes simply,

$$(12.13) \quad \bar{q} \hat{n}_1 = g \bar{h}$$

By performing basic L^2 -estimate and pressure estimate (we skip detail here)

$$(12.14) \quad \|\bar{v}(t)\|_{L^2}^2 + \|\bar{B}(t)\|_{L^2}^2 + |\bar{h}(t)|_{L^2}^2 \leq \Lambda(R) \int_0^t \left(\|\bar{v}(s)\|_{H^1}^2 + \|\bar{B}(s)\|_{H^1}^2 + |\bar{h}(s)|_{H^{\frac{1}{2}}}^2 \right) ds$$

We should control $\|v\|_1$ on right hand side. But, since there are no dissipation on the left hand side, we cannot make it absorbed. Instead, we use vorticity. Let's define vorticity $\omega_v = \nabla^\varphi \times v$, $\omega_B = \nabla^\varphi \times B$ (which is equivalent to $\omega_v = (\nabla \times u)(t, \Phi)$ and $\omega_B = (\nabla \times H)(t, \Phi)$). We have (same for B)

$$\begin{aligned} \omega_v \times \hat{n} &= \frac{1}{2} (D^\varphi v \hat{n} - (D^\varphi v)^T \hat{n}) \\ &= S^\varphi v \hat{n} - (D^\varphi v)^T \hat{n} = \frac{1}{2} \partial_{\hat{n}} u - g^{ij} (\partial_j v \cdot \hat{n}) \partial_{y^i} \end{aligned}$$

Hence, it is suffice to estimate ω_v instead of $\partial_z v$, *i.e*

$$(12.15) \quad \|\partial_z v\|_{L^2} + \|\partial_z B\|_{L^2} \leq \Lambda(R) \left(\|\omega_v\|_{L^2} + \|\omega_B\|_{L^2} + \|v\|_1 + |h|_{\frac{1}{2}} \right)$$

To estimate ω (both for v, B), we use vorticity equation.

$$(12.16) \quad (\partial_t^{\varphi_i} + v_i \cdot \nabla^{\varphi_i}) \omega_i = (\omega_i \cdot \nabla^{\varphi_i}) v_i$$

L^2 energy estimates of $\bar{\omega}_v$ and $\bar{\omega}_B$ are

$$(12.17) \quad \|\bar{\omega}_v(t)\|_{L^2}^2 + \|\bar{\omega}_B(t)\|_{L^2}^2 \leq \Lambda(R) \int_0^t \left(|\bar{h}(s)|_1^2 + \|\bar{v}(s)\|_{H^1(S)}^2 + \|\bar{\omega}_v(s)\|_{L^2}^2 + \|\bar{B}(s)\|_{H^1(S)}^2 + \|\bar{\omega}_B(s)\|_{L^2}^2 \right) ds$$

So we finish the proof.

13. INVISCID LIMIT

This part is exactly same as [1]. For interior, it is clear that we can just add B -related terms those have same regularity as v . For boundary condition, since we have $B = 0$ on the boundary, there is nothing to worry about weak sense definition. At result, we get sequence (up to subsequence) $(v^\varepsilon(t), B^\varepsilon(t), h^\varepsilon(t))$ converge strongly in L^2 to its weak limit (v, B, h) in $L^2(S) \times L^2(S) \times L^2(\mathbb{R}^2)$. L^∞ follows from L^2 convergence, uniform bounds of energy, and anisotropic embedding.

14. SURFACE TENSION CASE

Another similar problem is surface tension case. In [23], surface tension case was solved when we only consider Navier-Stokes equation, not MHD. Main difference between [1] between [23] is the regularity of h . Surface tension improves the regularity of h by 1 than without surface tension case. At result, we don't have to consider Alinhac's unknown, since no harmful commutators appear. But we loose $1/2$ regularity of h in pressure estimate. This problem was solved by taking time-derivatives in [23]. Nearly same strategy can be used to solve free boundary MHD. If we add surface tension, (1.4) becomes,

$$(14.1) \quad \begin{cases} \partial_t^\varphi v + v \cdot \nabla^\varphi v - B \cdot \nabla^\varphi B + \nabla^\varphi q = 2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi v), & \text{in } S \\ \partial_t^\varphi B + v \cdot \nabla^\varphi B - B \cdot \nabla^\varphi v = 2\varepsilon \nabla^\varphi \cdot (\mathbf{S}^\varphi B), & \text{in } S \\ \nabla^\varphi \cdot v = 0, & \text{in } S \\ \nabla^\varphi \cdot B = 0, & \text{in } S \\ q^b \mathbf{n} - 2\varepsilon (\mathbf{S}^\varphi v) \mathbf{n} = g h \mathbf{n} - \eta \nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}, & \text{on } \partial S \\ \partial_t h = v \cdot \mathbf{N}, & \text{on } \partial S \\ B = 0, & \text{on } \partial S \\ v(0) = v_0, \quad B(0) = B_0, & \text{in } S \end{cases}$$

where η is surface tension constant and with compatibility condition

$$(14.2) \quad \begin{cases} \nabla^\varphi \cdot v_0 = 0, & \text{in } S \\ \nabla^\varphi \cdot B_0 = 0, & \text{in } S \\ B_0 = 0, & \text{on } \partial S \\ \Pi S^\varphi(v_0)\mathbf{n} = 0, & \text{on } \partial S \end{cases}$$

As we can see in [23], we need only standard estimates. (We don't have to estimate L_T^4 type estimate.) This is also true for MHD-surface tension case. By the cancellation between the Lorentz force term in Navier-Stokes and the source term in Faraday's law, nonlinear terms on right hand side do not generate any problematic estimate. At result, we will get the following regularity.

$$v, B \in L^\infty H_{co}^m \cap W^{1,\infty} H_{co}^{m-1} \cap \dots W^{m-1,\infty} H_{co}^1 \cap W^{m,\infty} L^2$$

We briefly explain main idea for how close energy estimate on sobolev conormal space.

14.1. Energy estimate of \mathbf{v} and \mathbf{h} . Applying $Z_x^m = \partial_x^{\alpha_x} \partial_y^{\alpha_y} \left(\frac{z}{1-z} \partial_z \right)^{\alpha_z}$, then our estimate for Navier-Stokes looks like,

$$(14.3) \quad \begin{aligned} E_0 &\doteq \|v\|_{H_{co}^m}^2 + |h|_{H^{m+1}}^2 + \varepsilon \int_0^t \|\nabla v\|_{H_{co}^m}^2 + \varepsilon \int_0^t \|\nabla B\|_{H_{co}^m}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_0(s) + \|\nabla v\|_{H_{co}^{m-1}} + \|\nabla B\|_{H_{co}^{m-1}} + |h|_{H^{m+\frac{3}{2}}} \right) ds \end{aligned}$$

where, C_0 is some terms of initial conditions, R contains E_0 and some low order L^∞ -type terms. Note that $|h|_{H^{m+\frac{3}{2}}}$, on the right hand side comes from surface tension term. We cannot control this term by E_0 . To estimate $|h|_{H^{m+\frac{3}{2}}}$, we use Dirichlet-Neumann estimate. We decompose pressure so that q^S solves $v_t + (\nabla q^S) = 0$. Then, using

$$(14.4) \quad v_t^b + (\nabla q^S)^b = 0$$

and, by kinematic boundary condition,

$$h_{tt} = v_t^b \cdot N + v^b \cdot N_t$$

So,

$$(14.5) \quad h_{tt} = -(\nabla q^S)^b \cdot N + v^b \cdot N_t$$

Since $(q^S)^b \sim \Delta h$, we can get $h_{tt} \sim \nabla \Delta h$, so it seems like $\partial_t h \sim \partial_x^{\frac{3}{2}} h$. For higher order, then $|h|_{H^{m+\frac{3}{2}}} \sim |\partial_t h|_{H^m}$. Hence our next step energy estimate is gained by applying $\partial_t Z_x^{m-1}$

$$(14.6) \quad \begin{aligned} E_1 &\doteq \|\partial_t v\|_{H_{co}^{m-1}}^2 + |\partial_t h|_{H^m}^2 + \varepsilon \int_0^t \|\nabla \partial_t v\|_{H_{co}^{m-1}}^2 + \varepsilon \int_0^t \|\nabla \partial_t B\|_{H_{co}^{m-1}}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_1(s) + \|\nabla \partial_t v\|_{H_{co}^{m-2}} + \|\nabla \partial_t B\|_{H_{co}^{m-2}} + |\partial_t h|_{H^{m+\frac{1}{2}}} \right) ds \end{aligned}$$

where, in this case, R contains E_0 and E_1 and some low order L^∞ -type terms. Since E^1 contains $|\partial_t h|_{H^m}$, it controls $|h|_{H^{m+\frac{3}{2}}}$, the bad commutator in previous step energy estimate (1.30). We perform this process repeatedly, until E_m step, where we get

$$(14.7) \quad \begin{aligned} E_2 &\doteq \|\partial_t^2 v\|_{H_{co}^{m-2}}^2 + |\partial_t^2 h|_{H^{m-1}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^2 v\|_{H_{co}^{m-2}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^2 B\|_{H_{co}^{m-2}}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_2(s) + \|\nabla \partial_t^2 v\|_{H_{co}^{m-3}} + \|\nabla \partial_t^2 B\|_{H_{co}^{m-3}} + |\partial_t^2 h|_{H^{m-\frac{1}{2}}} \right) ds \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& \vdots \\
(14.8) \quad E_k & \doteq \|\partial_t^k v\|_{H_{co}^{m-k}}^2 + |\partial_t^k h|_{H^{m-k+1}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^k v\|_{H_{co}^{m-k}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^k B\|_{H_{co}^{m-k}}^2 \\
& \leq C_0 + \Lambda(R) \int_0^t \left(E_k(s) + \|\nabla \partial_t^k v\|_{H_{co}^{m-k-1}}^2 + \|\nabla \partial_t^k B\|_{H_{co}^{m-k-1}}^2 + |\partial_t^k h|_{H^{m-k+\frac{3}{2}}}^2 \right) ds, \quad 1 \leq k \leq m-1
\end{aligned}$$

$$\begin{aligned}
& \vdots \\
& \vdots \\
(14.9) \quad E_{m-1} & \doteq \|\partial_t^{m-1} v\|_{H_{co}^1}^2 + |\partial_t^{m-1} h|_{H^2}^2 + \varepsilon \int_0^t \|\nabla \partial_t^{m-1} v\|_{H_{co}^1}^2 + \varepsilon \int_0^t \|\nabla \partial_t^{m-1} B\|_{H_{co}^1}^2 \\
& \leq C_0 + \Lambda(R) \int_0^t \left(E_{m-1}(s) + \|\nabla \partial_t^{m-1} v\|_{H_{co}^1}^2 + \|\nabla \partial_t^{m-1} B\|_{H_{co}^1}^2 + |\partial_t^{m-1} h|_{H^{\frac{5}{2}}}^2 \right) ds
\end{aligned}$$

If we sum all above m estimates, then $E_0 + E_1 + \dots + E_{m-2} + E_{m-1}$ controls every high order terms of h except $|\partial_t^{m-1} h|_{H^{\frac{5}{2}}}$.

14.2. Energy estimate for all-time derivatives. In the last step, if we take only time derivatives, energy becomes

$$E^m \doteq \|\partial_t^m v\|_{L^2}^2 + |\partial_t^m h|_{H^1}^2 + \varepsilon \int_0^t \|\nabla \partial_t^m v\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla \partial_t^m B\|_{L^2}^2$$

Our claim is that energy estimate is closed by E^m , that means $|\partial_t^m h|_{H^{3/2}}$ does not appear on the right hand side. Details for this part is nearly same as [23], because source terms (on the right hand side of two PDEs) are canceled by each other by standard estimate. Bad terms which look like $|\partial_t^m h|_{\frac{3}{2}}$ and way of controlling them are exactly same as [23].

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